On Shape Delaunay Tessellations*

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Abstract

Shape Delaunay tessellations are a generalization of the classical Delaunay triangulation of a finite set of points in the plane, where the empty circle condition is replaced by emptiness of an arbitrary convex compact shape. We present some new and basic properties of shape Delaunay tessellations, concerning flipping, subgraph structures, and recognition.

Keywords: Delaunay triangulation; convex distance function; flipping; proximity graphs; recognition

1 Introduction

The well-known Delaunay triangulation (see, for example, [4, 9]) of a set \( S \) of \( n \) point sites in the plane \( \mathbb{R}^2 \) can be defined in two ways: As the geometric dual of the Voronoi diagram of \( S \), when we connect two sites in \( S \) by a dual edge iff their Voronoi regions are edge-adjacent; or by using the empty circle property: An edge \( p\overline{q} \) between two sites \( p, q \in S \) is a Delaunay edge iff there exists some circle that passes through \( p \) and \( q \) but does not enclose any other site in \( S \).

For the Euclidean distance, these two definitions are equivalent (if \( S \) is in general position). However, for general convex distance functions, where the circular unit disk is replaced by a convex compact set \( C \) in \( \mathbb{R}^2 \) and some interior point as its center, this is no longer true. The reason is that the symmetry property is not guaranteed. More specifically, in the definition of the Voronoi diagram \( V_{dc}(S) \) for the convex distance function \( d_C \) based on shape \( C \), distances from the sites are measured (see Chew and Drysdale [7]), whereas for the empty circle property, distances are taken from the center of \( C \), i.e., towards the sites. Since we have \( d_C(p, q) = d_C(q, p) \), where \( C' \) denotes the reflected image of \( C \) about its center, the dual of \( V_{dc}(S) \) is just the ‘empty-shape’ Delaunay tessellation for the shape \( C' \) which does not depend on the location of the shape center. This fact has an interesting consequence: The dual of \( V_{dc}(S) \), and thus, the combinatorial structure of \( V_{dc}(S) \), is invariant under movements of the center of \( C \). The geometric structure of \( V_{dc}(S) \) changes, of course.

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Figure 1: A shape Delaunay tessellation where \( C \) is an obtuse triangle (whose homothetic copies are shown lightly shaded). The support hull is emphasized. Note the direction-sensitivity of the tessellation’s triangles according to \( C \).

Drysdale [8] considered such so-called shape Delaunay tessellations, and showed how the divide & conquer approach in Guibas and Stolfi [10] can be extended to run, in optimal time \( O(n \log n) \), for general convex distance functions. Ma [13] proposed an incremental algorithm that restores the tessellation after the insertion of each site. Skyum [15] constructs the shape Delaunay tessellation, based on a smooth and strictly convex distance function, using the sweep line approach, in \( O(n \log n) \) many steps.

To give a precise (edge-based) definition, an edge \( p\overline{q} \) connecting two sites \( p, q \in S \) belongs to the shape Delaunay tessellation, \( DT_C(S) \), iff there exists a homothet of \( C \) with \( p \) and \( q \) on its boundary and covering no other site in \( S \). (A homothet of \( C \) is a set that can be obtained from \( C \) by translation and scaling.) If \( C \) is not smooth, then \( DT_C(S) \) need not be a full triangulation of the convex hull of \( S \). Consult Figure 1: Edge \( e \) is not part of \( DT_C(S) \), because any homothet of the triangular shape \( C' \) with \( e \) as a chord will contain some other site. The subset of the plane covered by \( DT_C(S) \) is called the support hull of \( S \) (with respect to the distance function \( d_C \) in [8]), generalizing the hull concept in Lee [12] for the \( L_p \)-metrics. The support hull is connected, as it has to contain a certain spanning tree of \( DT_C(S) \), see later, and \( DT_C(S) \) is a full triangulation therein.

Shape Delaunay tessellations have several interesting applications. The possible asymmetry of \( C \) carries over to the shape of the triangles in \( DT_C(S) \), useful in designing triangular networks of specific direction-
sensitivity. Also, high-quality spanner graphs can be extracted, for instance when \( C \) is an equilateral triangle; see Chew [6]. In fact, the dilation (i.e., the factor lost when following shortest paths in \( \text{DT}_C(S) \) rather than going straight) is a constant depending only on \( C \), as Bose et al. [5] have shown. For certain shapes \( C \), \( \text{DT}_C(S) \) undergoes only a nearly-quadric number of discrete changes when the sites in \( S \) are moved on trajectories of constant description. See Abam et al. [1] where \( C \) is a sharp diamond shape, and Agarwal et al. [2] where \( C \) is a small regular polygon.

In the present note, we will elaborate on some basic questions about shape Delaunay tessellations which seem to have received less attention so far. In fact, things well known for the classical Delaunay triangulation, like flipping, subgraph structures, and recognition, get more involved for general shapes \( C \) and lead to interesting questions. Part of the presented material can be found in more detail in Paulinì [14].

We remark that if the definition of \( \text{DT}_C(S) \) were based on its triangles rather than on its edges (which is equivalent for the case of \( C \) being a circular disk), then more edges were possibly included, dual to unbounded parallel edges in the Voronoi diagram \( V_{d_C}(S) \). In other words, the corresponding support hull could enlarge.

2 Flipping

Flipping edges in shape Delaunay tessellations has been considered in Ma [13], as the dual process of inserting a Voronoi region in the diagram \( V_{d_C}(S) \). Here we will deal with the problem of transforming an arbitrary triangulation of \( S \) into the shape Delaunay tessellation \( \text{DT}_C(S) \), where the existence of a respective sequence of Delaunay flips is less obvious.

In principle, the empty shape property can be used to decide ‘Delaunayhood’ of an edge, but some care has to be taken. Let \( e \) be any edge spanned by the point set \( S \). For the given convex shape \( C \), consider its homothetic copy of infinite size, and denote with \( C_L \) and \( C_R \) its two translates having the endpoints of \( e \) on the boundary, from the left and the right side, respectively. (Without loss of generality, we assume that edge \( e \) is not horizontal.) We distinguish three disjoint (open) domains in \( \mathbb{R}^2 \) as shown in Figure 2.

\[
I = C_L \cap C_R, \quad II = \mathbb{R}^2 \setminus (C_L \cup C_R),
\]

and

\[
III = (C_L \cup C_R) \setminus (C_L \cap C_R).
\]

Clearly, if the shape \( C \) is smooth (for example, a circular disk, as for the classical Delaunay triangulation), then \( C_L \) and \( C_R \) are just the two open half-planes bounded by the straight line \( \ell \) supporting \( e \), and the distinction above is not meaningful, because \( I = II = \emptyset \), and \( III = \mathbb{R}^2 \setminus \ell \) is the only domain.

For the non-smooth case, if domain \( I \) contains any site from \( S \), then edge \( e \) obviously cannot be an edge of \( \text{DT}_C(S) \). We call \( e \) an invalid edge in this case. Also, no site in domain \( II \) can form a valid triangle with edge \( e \), whereas a site \( p \) in domain \( III \) can do so, because some homothet of \( C \) which covers only \( e \) and \( p \) might exist. In the case where \( S \), apart from \( e \)’s endpoints, lies entirely within the domain \( II \), edge \( e \) is part of the support hull of \( S \), from both sides. Indeed, this situation might occur for all valid edges, and \( \text{DT}_C(S) \) then degenerates to a tree, containing no valid triangles.

It is well known that any given triangulation \( T \) of \( S \) can be transformed into the (classical) Delaunay triangulation \( \text{DT}(S) \) of \( S \), by applying a sequence of edge flips based on the ‘local Delaunayhood’ of an edge. Local Delaunayhood is defined via the empty circumcircle property, but can be equivalently characterized by an angle criterion.

More specifically, consider any convex quadrangle \( Q \), given as the union of two triangles \( \Delta_i \) and \( \Delta_j \) in the triangulation \( T \). These triangles share an edge \( e \). We flip \( e \) by replacing \( e \) by the other diagonal of \( Q \), and consider the two triangles that \( Q \) is split into now. Then, if \( e \) was not locally Delaunay, the circumcircles of \( \Delta_i \) and \( \Delta_j \) each enclosed some vertex of \( Q \), whereas the circumcircles of the two new triangles do not; see Figure 3. Equivalently, the smallest angle arising in the former triangles is smaller than the smallest angle arising in the new triangles. That is, the angularity of \( T \) increases lexicographically. This property is, unfortunately but not surprisingly, lost for general convex shapes \( C \), and a related criterion that maintains the analogy between \( \text{DT}(S) \) and \( \text{DT}_C(S) \) is of interest, not least for algorithmic purposes.

Indeed, assuming that the \( \Delta_1/\Delta_2 \) partition of \( Q \) is locally Delaunay and the \( \Delta_3/\Delta_4 \) partition is not, for \( \text{DT}(S) \) we also have the inequalities

\[
\begin{align*}
\max\{r_1, r_2\} &< \max\{r_3, r_4\} \quad (1) \\
\min\{r_1, r_2\} &< \min\{r_3, r_4\} \quad (2)
\end{align*}
\]
Let us first assume that $C$ is smooth. Then the circum-shape, $C(\Delta)$, of a triangle $\Delta$, that is, the homothet of $C$ that has the three vertices of $\Delta$ on its boundary, always exists.

By contrast, the smallest enclosing shape, $M_\Delta$, of $\Delta$, which is the smallest homothet of $C$ that covers $\Delta$, also exists when $C$ is not smooth. Note that these two concepts are different already when $C$ is a circular disk, and an obtuse triangle is taken for $\Delta$. $C(\Delta)$ and $M_\Delta$ need not be unique if $C$ fails to be strictly convex, however. See e.g. Alonso et al. [3] who discuss this and related issues in detail.

For the convex quadrangle $Q$, we call a side $e$ of $Q$ (locally) Gabriel with respect to $C$ if every smallest enclosing shape $M_e$ of $e$ is empty of other vertices from $Q$.

**Lemma 1** Let $C$ be a smooth convex shape, and suppose that the $\Delta_1/\Delta_2$ partition of $Q$ is locally Delaunay with respect to emptiness of $C(\Delta_1)$, whereas the $\Delta_3/\Delta_4$ partition is not. Further, let $r_i$ denote the scaling factor of $C(\Delta_i)$.

(a) If $Q$ has two opposite Gabriel sides then both inequalities (1) and (2) hold.

(b) Otherwise, $Q$ must have two adjacent Gabriel sides, and the max-inequality (1) still holds.

**Proof.** (a) Any smallest shape $M_e$ covering a Gabriel side $e$ is free of other vertices from $Q$. When letting $M_e$ grow, the first vertex reached will build a Delaunay triangle, with smaller circum-shape compared to the triangle obtained from the second vertex. This means $r_1 < r_3$ (without loss of generality). Likewise, starting from the opposite Gabriel side we get $r_2 < r_4$, and the inequalities (1) and (2) are both fulfilled.

(b) Without two opposite Gabriel sides there have to be two adjacent non-Gabriel sides, $ab$ and $bc$, with a vertex $d$ of $Q$ inside both of $M_{ab}$ and $M_{bc}$. (This can never happen for circular shapes, because the relevant boundary point of $M_{ab} \cap M_{bc}$ then lies on the line through $a$ and $c$ by Thales’ theorem, so that $d$ cannot belong to a convex quadrangle.)

Consult Figure 4. From $d \in M_{ab} \cap M_{bc}$ it follows that $d \in M_{ab}$. Moreover, none of the other vertices can be inside $M_{ab}$ or $M_{bc}$, because for the respective scaling factors $f_i$ of the shapes we have $f_{ab} < f_{bc}$, $f_{ab}$, and $f_{bc} < f_{bc}$, $f_{bc}$. So $ad$ and $cd$ are both Gabriel sides, which can build their Delaunay triangles $\Delta_1$ and $\Delta_2$ with $\max\{r_1, r_2\} < r_4$. □

As a corollary, every convex quadrangle has at least two (local) Gabriel sides, for any given convex shape $C$.

If the shape $C$ is not smooth, $DT_C(S)$ is not a full triangulation in general. Invalid edges may be involved in the flipping procedure, where there is a vertex of $Q$ inside the respective domain $I$ in Figure 2. In particular, for a triangle $\Delta_i$ there may be no homothet of $C$ that has the vertices of $\Delta_i$ on its boundary, i.e., $C(\Delta_i)$ does not exist. In this case, we define the scaling factor to be infinite.

With infinite scaling factors, we may have situations where $\max\{r_1, r_2\} = \infty$ and $\max\{r_3, r_4\} = \infty$, so inequality (1) is insufficient and needs to be extended to a more general form.

**Theorem 2** Let the quadrangle $Q$, its triangles $\Delta_i$, and the scaling factors $r_i$ be defined as in Lemma 1. Moreover, put an indicator $D_{i,j}$ to 1 if the diagonal between $\Delta_i$ and $\Delta_j$ is an invalid edge, and to 0, otherwise. The following lexicographical inequality holds for arbitrary convex shapes $C$:

\[
\begin{pmatrix}
\max\{r_1, r_2\} \\
\min\{r_1, r_2\}
\end{pmatrix}
\begin{pmatrix}
D_{1,2}
\end{pmatrix}
<
\begin{pmatrix}
\max\{r_3, r_4\} \\
\min\{r_3, r_4\}
\end{pmatrix}
\begin{pmatrix}
D_{3,4}
\end{pmatrix}
\]

**Proof.** If $\max\{r_1, r_2\} < \infty$ then inequality (1) holds by Lemma 1. Otherwise, if $\min\{r_1, r_2\} < \infty$ then inequality (2) holds, because the special case where $\min\{r_1, r_2\} > \max\{r_3, r_4\}$ (in Figure 4) cannot occur with either $r_1$ or $r_2$ being infinite. If both inequalities fail, we compare the diagonal edges. Obviously an invalid edge cannot be locally Delaunay; see also Figure 5. □
Let \( T \) be a triangulation that is locally shape Delaunay but different from \( \text{DT}_C(S) \). Then \( T \) contains at least one triangle \( \Delta \) with an edge \( e \) and a vertex \( q \), so that \( q \in C(\Delta) \) and \( q \) sees \( e \) by the largest angle; see Figure 6.

Let \( \Delta' \) be the triangle adjacent to \( \Delta \) at \( e \), with edge \( e' \) towards \( q \), and \( p \) as new vertex. By the local Delaunay property, \( p \) lies outside \( C(\Delta) \). On the other hand, because \( C \) is convex, \( q \) lies inside \( C(\Delta') \) and sees \( e' \) by a larger angle than \( e \), a contradiction. \( \square \)

Compare also Figure 7 for the non-smooth case, which complements Figure 6. There is a special case if \( C \) is not strictly convex, as has been observed in Drysdale [8]; see Figure 8. Vertex \( p \) from above may lie on the boundary of \( C \), yielding a ‘cocircularity’. However, this case can easily be resolved when flipping is based on inequality (2) for the scaling factors.

Theorem 3 together with Theorem 2 give the following optimization result: \( \text{DT}_C(S) \) lexicographically minimizes the decreasingly sorted vector of scaling factors of the triangles’ circum-shapes, for all possible triangulations of the point set \( S \). In particular, the largest arising circum-shape is minimized.

As a related problem, Lambert [11] elaborated on the question about which flip operations, based on a given set of convex shapes, guarantee a unique locally optimal triangulation, i.e., one where no Delaunay flips do exist. He proved that the underlying set of shapes has to be a homothetic family, as it is the case for \( \text{DT}_C(S) \).

The algorithmic question of how many Delaunay flips are required to transform a given triangulation \( T \) of \( S \) into \( \text{DT}_C(S) \) possibly depends on the choice of \( C \) and remains open. Of course, the lower bound of \( \Omega(n^2) \) for the classical Delaunay triangulation (see, e.g., [9]) carries over, if \( C \) can be any convex shape.

3 Subgraphs

The Delaunay triangulation is not only useful because its triangles optimize angles and circumradii (and
other quantities), but also because it contains various subgraphs which arise in different applications. The minimum spanning tree, the relative neighborhood graph, and the Gabriel graph are among these subgraphs; see e.g. [4] for an overview. When considering whether such structures extend to general convex distance functions \(d_C\), we encounter two problems:

First of all, the function \(d_C\) is not a metric unless the shape \(C\) is point-symmetric. Also, \(d_C\) depends on the chosen center \(x\) of \(C\), whereas the triangulation \(DT_C(S)\) does not. Still, a convex distance function can be constructed that is center-independent and symmetric, and which yields the desired subgraph properties, as we describe next.

Assume that \(C\) is centered at the origin, and denote with \(C_x\), for \(x \in C\), the translate of \(C\) by \(x\). Given two points \(p\) and \(q\), we define the distance function

\[ m_C(p, q) = \min_{x \in C} d_C(x, p, q). \]

Observe that \(m_C(p, q)\) equals the Euclidean distance \(d(p, q)\), divided by the length of the longest line segment contained in \(C\) and parallel to the edge \(pq\). Thus the concept of the, in general, non-symmetric radius is replaced by the concept of symmetric diameter. In fact, \(m_C(p, q)\) is just the scaling factor \(f_{pq}\) of the smallest homothet of \(C\) that covers \(pq\) (used in the proof of Lemma 1). To see that we really deal with a convex distance function, observe that

\[ m_C(p, q) = d_{\hat{C}}(p, q), \text{ with } \hat{C} = \bigcup_{x \in C} C_x. \]

The symmetric convex shape \(\hat{C}\) is the Minkowski sum of \(C\) and its reflected shape. (The Minkowski sum of two sets \(A\) and \(B\) in the plane is defined as \(A \oplus B = \{a + b \mid a \in A, \ b \in B\}\). Loosely speaking, \(A \oplus B\) is the union of all translates of \(B\) that overlap with \(A\).)

Coming back to the question of subgraphs, we can now use the following facts: For any symmetric shape, and in particular for \(\hat{C}\), the equality \(m_{\hat{C}}(p, q) = \frac{1}{2} d_{\hat{C}}(p, q)\) holds, such that the two metrics \(m_{\hat{C}}\) and \(d_{\hat{C}}\) yield the same minimum spanning tree. Further, we have \(m_{\hat{C}}(p, q) = \frac{1}{2} m_C(p, q)\) for general convex shapes \(C\). In conclusion, the minimum spanning tree with respect to the metric \(m_{\hat{C}}\), which is contained in \(DT_{\hat{C}}(S)\) because \(m_{\hat{C}}\) is symmetric, is the minimum spanning tree of \(S\) with respect to \(m_C\). This tree is indeed a subgraph of the original shape Delaunay tessellation, \(DT_C(S)\), by Theorem 6 below. We obtain the following structural result:

**Theorem 4** Let \(C\) be an arbitrary convex shape. The minimum spanning tree of the point set \(S\) with respect to \(m_C\) is a subgraph of both tessellations \(DT_C(S)\) and \(DT_{\hat{C}}(S)\).
sites \( r \in S \). (Again, this definition is conformal with the classical case of a circular disk.) When letting \( \text{MST}_C(S) \) denote the minimum spanning tree of \( S \) with respect to the metric \( m_C \), the following inclusion relations hold, in generalization of the well-known classical case.

**Theorem 6** For any given convex shape \( C \), we have \( \text{MST}_C(S) \subseteq \text{RNG}_C(S) \subseteq \text{GG}_C(S) \subseteq \text{DT}_C(S) \).

**Proof.** Assuming there is an edge \( \overline{pq} \) in \( \text{MST}_C(S) \) that is not in \( \text{RNG}_C(S) \), there has to be a site \( r \) with \( \max\{m_C(p,r),m_C(q,r)\} < m_C(p,q) \). By deleting edge \( \overline{pq} \) and inserting \( \overline{pr} \) or \( \overline{qr} \) (whichever reconnects the tree) one gets a shorter tree. This is a contradiction; see also Figure 11.

If there is an edge \( \overline{pq} \) in \( \text{RNG}_C(S) \) that is not in \( \text{GG}_C(S) \), there has to be some site \( r \) inside \( M_{\overline{pq}} \). It follows that \( m_C(p,r) < m_C(p,q) \) and \( m_C(q,r) < m_C(p,q) \), a contradiction.

Assuming there is an edge \( \overline{pq} \) that is in \( \text{GG}_C(S) \) but not in \( \text{DT}_C(S) \), there is no homothet of \( C \) which solely covers the sites \( p \) and \( q \), especially not \( M_{\overline{pq}} \), a contradiction again. \( \square \)

Figure 11 displays a point set \( S \) where, for a triangular shape \( C \), the second inclusion stated in Theorem 6 is proper. This should be compared with the result given in Lemma 5.

Concerning the symmetric Minkowski sum shape \( \hat{C} \), we have \( \text{MST}_{\hat{C}}(S) = \text{MST}_C(S) \) as shown before, and also \( \text{RNG}_{\hat{C}}(S) = \text{RNG}_C(S) \), but \( \text{GG}_{\hat{C}}(S) \neq \text{GG}_C(S) \) and \( \text{DT}_{\hat{C}}(S) \neq \text{DT}_C(S) \) in general, as is reflected in Figures 9 and 10.

4 Recognition

Recognizing whether a given triangulation \( T \) of the point set \( S \) is the Delaunay triangulation is an easy task, by exploiting the empty circle property: For each triangle of \( T \), one just checks if its circumcircle encloses any other point of \( S \). More interesting is the corresponding recognition problem for shape Delaunay tessellations: Given \( T \), does there exist a convex shape \( C \) such that \( T = \text{DT}_C(S) \) ?

There are triangulations of only 7 points which cannot be generated by any convex shape, like the one drawn in Figure 12.

Figure 12: To have edge \( e \) included in \( \text{DT}_C(S) \), the shape \( C \) has to be higher than wide. But edge \( e' \) requires \( C \) to be wider than high.

On the other hand, there exist point sets \( S \) that allow for an exponential number of different shape Delaunay tessellations. This is sketched in the construction in Figure 13. Let \( |S| = n \) be even. We choose points \( 1, 2, \ldots, n-1 \) of \( S \) to be almost cocircular, and such that \( \text{DT}(S) \) connects all of them to the last point, \( p \). The different shapes are circular disks, slightly truncated by straight lines. When cutting off slim caps with lines parallel to \( i-1, i+1, \) for
Figure 13: Classical Delaunay triangulation (left). Locally flattening the circular shape (right) causes a flip from edge \( i, p \) to edge \( i-1, i+1 \), for any even index \( i \). For example, taking \( i \) from the subset \( \{2, 4\} \) produces the shape Delaunay tessellation shown in the middle.

certain even indexed points \( i \), the obtained shape \( C \) will make point \( p \) lose connection in \( \text{DT}_C(S) \) to any possible subset of \( \{2, 4, 6, \ldots, n-2\} \). This gives rise to \( 2^{n/2-1} \) different triangulations.

While we conjecture the recognition problem to be NP-hard in its general form, a polynomial-time solution exists if \( C \) is assumed to be an (unknown) convex polygon with a constant number of sides.

Let us suppose first that \( C \) is of triangular shape. We are looking for all triangles \( \Delta \) for which \( T = \text{DT}_\Delta(S) \) holds. To be precise, \( T \) need not be a full triangulation of \( S \), but rather of the support hull, \( H_T \), which we assume to be given as well. We restrict the possible form of \( \Delta \) by looking at the individual components of \( T \).

If \( H_T \) has some boundary vertex of exterior angle \( \alpha < \pi \), then \( \Delta \) has to contain some vertex with angle \( \beta < \alpha \) which 'fits in', compare Figure 1: An edge \( e \) outside of \( H_T \) would exists, otherwise, such that \( e \) can be covered by an infinite copy of \( \Delta \) which contains no other point of \( S \), and \( e \) would belong to \( \text{DT}_\Delta(S) \), a contradiction. Combining all angle possibilities yields a candidate for \( \Delta \) that conforms with \( H_T \), or already shows that none exists. As two angle restrictions might belong to the same vertex, or to two different vertices of \( \Delta \), this takes \( O(n^2) \) time.

In addition to this ‘positive’ information from the support hull, each triangle, \( t \), of \( T \) yields ‘negative’ information, in the sense that \( \Delta \) must not fit in with any of its angles at a vertex \( v \) of \( t \). Otherwise, no homothet of \( \Delta \) can cover the edge \( e \) of \( t \) opposite to \( v \) without covering \( v \) as well, and \( e \) would not be part of \( \text{DT}_\Delta(S) \); see Figure 14. This condition guarantees that each edge \( e \) of \( t \) is locally valid with respect to \( t \). In other words, \( t \) then admits a finite circum-shape with respect to \( \Delta \); see Section 2.

Interestingly, because we restricted attention to triangular shapes, this already implies that \( e \) is an edge of \( \text{DT}_\Delta(S) \). Consider the smallest homothet \( \Delta_t \) of \( \Delta \) that covers \( t \) (that is, the smallest enclosing shape). Then \( \Delta_t \) cannot enclose any other point \( p \) in \( S \). Otherwise, there exists a triangle \( t' \) adjacent to \( t \) and with vertex \( p \), where the negative angle condition is violated by \( \Delta_t \); see Figure 14 again.

Algorithmically, the second condition yields \( O(n) \) angle intervals for each of the 3 vertices of \( \Delta \). This leads to a runtime of \( O(n^3) \), which can be reduced to \( O(n^2) \) by combining intervals additively (rather than multiplicatively) for two vertices via their connecting edge.

**Theorem 7** For a given (possibly partial) triangulation \( T \) of a set \( S \) of \( n \) points, it can be decided in \( O(n^2) \) time whether there is some triangular shape \( \Delta \) with \( \text{DT}_\Delta(S) = T \). In the affirmative case, a description of all possible candidates for \( \Delta \) is computed.

Note that the obvious necessary conditions for the corresponding support hull \( H_T \), namely, connected-
ness and absence of non-triangular 'holes' (see Section 1), can be checked beforehand in $O(n)$ time.

For convex polygonal shapes $C$ with $k = O(1)$ sides, a candidate can be found, if it exists, by checking polynomially many, $O(n^k)$, possibilities for angles at its vertices, in a way similar as described above.

References


