A note on visibility-constrained Voronoi diagrams

F. Aurenhammer, B. Su, Y.-F. Xu, B. Zhu

1. Introduction

The Voronoi diagram of a set S of n point sites in the plane is a well-known geometric data structure. It is a planar straight-line graph of size \(O(n)\) that associates with each site \(p \in S\) the convex region of all points in the plane closest to \(p\), under the Euclidean distance.

Among the numerous variations of this concept considered in the literature [4] are so-called visibility-constrained Voronoi diagrams. Here, a set \(B\) of opaque obstacles is given, in addition to the set \(S\) of sites. A site \(p\) and a point \(x\) in the plane can ‘see’ each other if the line segment \(px\) avoids \(B\), that is, if visibility between \(p\) and \(x\) is not blocked by any object in \(B\). The Voronoi region of \(p\) now consists of all points in the plane for which \(p\) is the closest visible site. Visibility-constrained Voronoi diagrams should not be confused with the concept of geodesic Voronoi diagrams, where shortest obstacle-avoiding paths are taken to measure distances [2,20,16].

Usually, the obstacle set \(B\) is modeled by \(m\) non-intersecting (open) line segments. There are two quite different cases, depending on whether or not these segments have their endpoints included in \(S\). The former case is easier and well-studied. It leads to the bounded Voronoi diagram considered in Wang and Schubert [24], where regions are (partially) clipped by the constraining segments and are nonconvex for sites that are segment endpoints; see Fig. 1(a). The geometric dual of this Voronoi diagram can be completed to what is called the constrained Delaunay triangulation of \(S\) and \(B\), see Fig. 1(b), a structure introduced in Lee and Lin [19] and studied by many authors [8,24,21,18]. Both structures are of linear size, \(O(n)\), and can be computed in optimal time \(O(n \log n)\). Note that we have \(|B| = m = O(n)|\), because \(B\) forms a planar graph whose vertices are from \(S\), and thus can consist of at most \(3n - 6\) segments.
In the latter case, where $B$ has no relation to $S$, the constraining line segments in $B$ do not exert proximity influence but just block visibility. Already two segments are enough to produce a diagram of size as large as $\Theta(n^2)$. For example, the so-called peeper’s Voronoi diagram achieves this bound; see Aurenhammer and Stöckl [5]. There visibility for each site $p \in S$ is constrained to the angle under which $p$ sees through a given ‘window’ between two collinear segments. The general case of $m$ segments among $n$ sites leads to a maximal diagram size of $\Theta(n^2 m^2)$ and has been treated in Wang and Tsin [25], also for the case where distances to the sites are weighted multiplicatively.

In this note we consider a modification where the diagram becomes significantly subquadratic in size and thus retains its usefulness in practical applications. In particular, the modification enables us to apply an interesting combinatorial theorem on arrangements of straight lines in the plane.

2. Baseline Voronoi diagram

Let us have a closer look at the case where visibility of each site $p_i \in S$ is constrained to an individual visibility angle, $\alpha_i$, at $p_i$.

The resulting Voronoi diagram is structured by $2n$ rays that bound the given $n$ wedges of visibility, with two rays emanating from each site $p_i$. This arrangement of wedges is a dissection of the plane into polygonal, not necessarily convex, cells. Each cell possibly gets further partitioned according to the nearest neighbor rule, by those sites which see the cell. That is, the plane is divided into maximal cells $C(T)$, exclusively seen by subsets $T \subseteq S$, and partitioned by their (classical) Voronoi diagram $V(T)$. Formally, the region of a site $p$ can be expressed as

$$\bigcup_{T \ni p} (C(T) \cap VR(p, T))$$

where $VR(p, T)$ is the region of $p$ in $V(T)$. See Fig. 2, where the shaded areas display the connected components of the region of site 4. Cells not visible from $S$, like the bottommost cell $A(\emptyset)$, belong to no site.

This concept covers the case where each site sees a half-plane (that is, where $\alpha_i = \pi$ for all $i$), considered earlier under the names semi Voronoi diagram in Cheng et al. [6] and half-plane Voronoi diagram in Fan et al. [13].

Regions in a Voronoi diagram for visibility angles are highly disconnected in general. Still, the total number of two-dimensional faces the plane gets split into is bounded by $O(n^2)$, as Fan et al. [14] have shown. This is somewhat surprising, as there may be quadratically many arrangement cells, with each cell containing a Voronoi diagram of potentially linear size. On the other hand, a size of $\Theta(n^2)$ does occur in the worst case, even if half-planes are taken as visibility wedges.

We now investigate a related half-plane-constrained yet combinatorially smaller diagram, which we shall call the baseline Voronoi diagram and which has been introduced in Su et al. [22]. The modification is simple but, as will be shown, of drastical effect.

Let a straight line $L_i$ be given through each site $p_i \in S$, termed its baseline. Denote with $H_i$ a fixed half-plane bounded by $L_i$. In the induced baseline Voronoi diagram, a point $x$ belongs to the region, $\text{reg}(p_i)$, of site $p_i$ if the following three conditions are satisfied:

1. $x \in H_i \quad (p_i \text{ sees } x)$
2. $|p_ix| \leq |p_jx|$, for all $p_j \in H_i \quad (p_i \text{ is the closest site that sees } x)$
3. $p_ix \cap L_k = \emptyset$, for all $k \neq i \quad (\text{view from } p_i \text{ to } x \text{ is not crossed by any baseline}).$

Regions can be defined in an equivalent and geometrically more intuitive way. For $L = \{L_1, \ldots, L_n\}$, denote with $A(L)$ the arrangement induced by these straight lines in the plane. Its cells are convex polygons. By condition (3) above, a site $p_i$ can only claim points for its region that lie within a cell of $A(L)$ with $p_i$ on its boundary. This cell, $C_i$, is unique by condition (1).
Fig. 2. Voronoi diagram constrained by angular visibility.

Fig. 3. Baseline Voronoi diagram for eight sites. Half-planes $H_i$ are indicated by arrows. One of the arrangement cells is shared by four sites and another one by two sites. Two more cells are owned by single sites. The area belonging to no site is left blank.

Note that $C_i = C_j$ is possible for $i \neq j$. Furthermore, by condition (2), $p_i$ must be the closest one among the set $S_i \subseteq S$ of sites situated on $C_i$'s boundary. In conclusion, we have

$$\text{reg}(p_i) = C_i \cap \text{VR}(p_i, S_i)$$

where $\text{VR}(p_i, S_i)$ is the region of $p_i$ in the classical Voronoi diagram $V(S_i)$ of $S_i$.

The baseline Voronoi diagram thus is composed of at most $n$ arrangement cells, each further refined by a Voronoi diagram. Its regions are, therefore, convex hence connected. Fig. 3 gives an illustration.

Intuitively speaking, for each site a 'safe' region of influence is obtained which is close to it. For example, sites could be fishing ships on fixed routes $L_i$. In order not to interfere with other routes, waters beyond the lines $L_i$ are not considered. Actions take place either port side or starboard side, indicated by the half-planes $H_i$.

The combinatorial complexity of this diagram is significantly subquadratic, though not linear, in the worst case. Its two-dimensional connected components will be called faces in the sequel.

**Theorem 1.** The baseline Voronoi diagram for $n$ point sites in the plane realizes $O(n)$ faces and $O(n^{4/3})$ edges and vertices.

**Proof.** Regions are nonempty and convex, and thus define exactly $n$ faces. These faces are pairwise interior-disjoint, and two faces can touch in at most one edge. Therefore, their complement in the plane (the area which belongs to no site) consists of only $O(n)$ additional faces.
Fig. 4. Modified baseline Voronoi diagram. Several sites share baselines. Some sites (□) are active on both sides, and others (●) only on one side.

Concerning edges, let $C_1, \ldots, C_k$, for $k \leq n$, be the arrangement cells appearing in the diagram. There are two types of diagram edges: border edges which refine the edges of $C_1, \ldots, C_k$, and bisector edges which stem from the Voronoi diagram $V(S_i)$ inside each cell $C_i$, defined by a subset $S_i$ of sites on its boundary. A single edge $e$ of $C_i$ might get split into several border edges, by Voronoï edges that are clipped to bisector edges by $e$; see Fig. 3.

As each site determines a unique cell, these cells induce a partition $S_1, \ldots, S_k$ of the set $S$ of sites. The corresponding Voronoi diagrams $V(S_i)$ are of linear complexity, $O(|S_i|)$. This implies that the total number of bisector edges is $O(\sum |S_i|) = O(n)$.

The number of border edges can be bounded by applying a result on the combinatorial complexity of arrangement cells. Clarkson et al. [9] proved that the total number of edges for any $k$ given cells in an arrangement of $n$ straight lines is $O(n^{2/3}k^{2/3} + n)$. This number does not increase in order when the edges of these cells are split into border edges by the $O(n)$ arising bisector edges. As we have $k \leq n$ in our case, this gives $O(n^{4/3})$ diagram edges in total. For vertices the bound is the same, because each vertex of the diagram is of degree at least 2.

Note that each subset $S_i$ above is in convex position: $S_i$ lies on the boundary of the convex polygon $C_i$, and thus each of its sites is a vertex of the convex hull of $S_i$. Hence the diagrams $V(S_i)$ are trees.

Note further that no two sites in $S_i$ lie on the same edge of $C_i$, which implies that the number of edges of $C_i$ is at least $|S_i|$. The asymptotic size of the baseline Voronoi diagram is, therefore, always determined by the complexity of its constituting arrangement cells, which is $O(n^{2/3}k^{2/3} + n)$ if there are $k$ cells. This bound is tight, by a corresponding lower bound given in Szemerédi and Trotter [23]. The maximal size of a baseline Voronoi diagram thus varies from $\Theta(n)$ for $k = 1$ to $\Theta(n^{4/3})$ for $k = n$.

In the former of these two extreme cases, a single cell is partitioned by the Voronoi diagram of all $n$ sites. On the other hand, there may be $n$ cells, each then empty of bisector edges. Note that, in this case, the diagram complexity might still be linear.

On the algorithmic side, the $k$ cells containing $n \geq k$ given points in an arrangement of $n$ lines can be computed in (roughly) $O(n^{4/3} \log n)$ time, by the algorithms developed in Edelsbrunner et al. [11] and Agarwal et al. [1], respectively.

To complete the desired diagram, for each cell $C_i$ the Voronoi diagram $V(S_i)$ it contains has to be constructed. As $S_i$ is in convex position, this can be done in $O(|S_i|)$ expected time, with the randomized incremental algorithm in Chew [7] which is easy to implement. Finally, the vertices of incidence between bisector edges and border edges have to be determined. To this end we merge, in linear time, the two cyclic sequences given by the edges of $C_i$ and the unbounded edges of $V(S_i)$, respectively. The total running time is not affected by these tasks.

There is a special situation where the baseline Voronoi diagram is always of small size.

**Theorem 2.** If all $n$ point sites are collinear then the combinatorial complexity of the baseline Voronoi diagram is $\Theta(n)$.

**Proof.** Assume that all the sites are placed on some straight line, $g$. The set of baselines, $L = \{L_1, \ldots, L_n\}$, is arbitrary. As we have $p_i = L_i \cap g$ for each site $p_i$, a cell of the arrangement $A(L)$ can serve as a cell $C_i$ for the baseline Voronoi diagram.
only if $g$ intersects it. The collection of the latter cells forms the so-called zone of $g$ in $\mathcal{A}(L)$, which is well known to be of complexity $\Theta(n)$; see e.g. Edelsbrunner [10]. The claimed size of the diagram follows. 

3. Extensions

Several modifications of the baseline Voronoi diagram are meaningful, for example where some (or all) of the $n$ sites exert influence on both sides of their baselines, or where several sites share a common baseline. In the former model, each site $p_i$ claims territory in the union of two neighbored cells in the arrangement $\mathcal{A}(L)$. This union is convex, as it represents a cell in the arrangement $\mathcal{A}(L \setminus \{L_i\})$. The region of $p_i$, however, fails to be convex in general, but it stays simply connected. The combinatorial complexity of $O(n^{4/3})$ and the construction time of $O(n^{4/3} \log n)$ remain unaffected for both models.

In Fig. 4, both modifications are combined. This models the situation where several ships follow the same route, and some of them consider ‘safe’ fishing grounds on both sides.

Generalizations of the baseline Voronoi diagram to higher dimensions are possible, for example, to 3-space where sites have associated ‘base-planes’ restricting their visibility. Regions are convex polyhedra now. Building on results in Edelsbrunner et al. [12], the maximum number of facets, edges, and vertices of the diagram is $\Theta(n^2)$. This complexity is already attained by $n$ given cells in an arrangement $\mathcal{A}$ of $n$ planes in 3-space. Observe that a possible subquadratic size of the constituting cells $C_i$ does not necessarily carry over to the base-plane Voronoi diagram, because the diagrams $V(S_i)$ inside these cells can be of size $\Theta(n^2)$ in 3-space as well. In fact, in a worst-case example where each pair of sites defines a Voronoi facet [15], the sites are in convex position, like in the sets $S_i$.

The known best complexity bound for the zone in $\mathcal{A}$ of a straight line $g$ (that is, for the collection of the $n + 1$ cells of $\mathcal{A}$ intersected by $g$) is still $O(n^2)$; see Houle and Tokuyama [17] and Aronov et al. [3]. It remains unclear whether there is an analogue of Theorem 2 for collinear (or more interestingly, coplanar) sites in dimension 3. In these special cases, the total size of the Voronoi diagrams $V(S_i)$ is only linear.

References


