**OPTIMAL TRIANGULATIONS**

**Introduction.** A *triangulation* of a given set $S$ of $n$ points in the Euclidean plane is a maximal set of non-crossing straight line segments (called *edges*) which have both endpoints in $S$. As an equivalent definition, a triangulation of $S$ is a partition of the convex hull of $S$ into triangular faces whose vertex set is exactly $S$. Triangulations are a versatile means for partitioning and/or connecting geometric objects. They are used in many areas of engineering and scientific applications such as finite element methods, approximation theory, numerical computation, computer-aided geometric design, computational geometry, etc. Many of their applications are surveyed in [8], [11], [20], [61].

A triangulation of $S$ can be viewed as a planar graph whose vertex set is $S$ and whose edge set is a subset of $S \times S$. The Eulerian relation for planar graphs implies that the number $e(S)$ of edges, and the number $t(S)$ of triangles, do not depend on the way of triangulating $S$. In particular

$$e(S) = 3n - 3 - h$$

$$t(S) = 2n - 2 - h$$

where $h$ denotes the number of edges bounding the convex hull of $S$. The number of different triangulations of $S$ is, however, an exponential function of $n$. More precisely, the number of triangulations every set of $n$ points (in general position) must have is $\Omega(2.63^n)$ [57]. A general upper bound is $O(43^n)$ [68], and point sets can be constructed that exhibit $\Theta(\sqrt{72}^n)$ triangulations [6]. All these bounds are very recent, and various prior bounds have been given; see the citations listed in [57, 68, 6].

The problem of automatically generating optimal triangulations for a point set $S$ has been a subject of research since decades. Enumerating all possible triangulations and selecting an optimal one (exhaustive search) is too time-consuming even for small $n$. In fact, constructing optimal triangulations in polynomial time is a challenging task. This becomes more apparent as greedy methods, such as deleting candidate triangles or edges from worst to best, are doomed to fail by the NP-completeness of the following problem; see Lloyd [56]: Given a point set $S$ and some set $E$ of edges on $S$, decide whether $E$ contains a triangulation of $S$.

Results on optimizing combinatorial properties of triangulations, such as their degree (Jansen [39]) or connectivity (Dey et al. [21] and Dillencourt [25]) are rare. Most optimization criteria for which efficient algorithms are known concern geometric properties of the edges and triangles.

**Delaunay triangulations.** The most commonly constructed and maybe the most famous triangulation for a point set $S$ is the Delaunay triangulation, $DT(S)$. See [8], [27], [34] [45] for extensive treatments and surveys. $DT(S)$ contains – for each triple of points in $S$ – the corresponding triangle provided its circumcircle is empty of points in $S$. Sibson [70] proved that $DT(S)$ can be constructed from any given triangulation $T$ of $S$ by applying any sequence of good edge flips. These are exchanges of diagonals in one of $T$’s convex quadrilaterals $Q$ such that – after the flip – the two new triangles are locally Delaunay, i.e., have circumcircles empty of vertices of $Q$.

Various global optimality properties of $DT(S)$ can be proved by observing that every good flip gives a local improvement of the respective optimality measure. Lawson [47] observed that equiangularity of a triangulation, which is the sorted list of its angles, increases lexicographically in this way. $DT(S)$ thus maximizes the minimum angle. Coarseness of a triangulation is measured by the largest circumcircle that arises for its triangles. D’Azevedo and Simpson [19] showed that $DT(S)$ minimizes coarseness in this sense, and also if smallest enclosing circles are taken rather than circumcircles. The latter property – unlike others – generalizes to higher-dimensional Delaunay triangulations; see Rajan [64]. Similarly, fatness may be defined as the...
sum of triangle inradii. Lambert [46] pointed out that \( DT(S) \) maximizes fatness or, equivalently, the mean inradius. Triangular surfaces obtained from lifting \( DT(S) \) to 3-space (for any given heights at triangle vertices) minimize \( \text{roughness} \), which is the integral of the squared gradient; see Rippa [66]. It is also known that a variant of \( DT(S) \) minimizes the minimum angle; see Eppstein [31].

The Delaunay triangulation is a special instance of regular triangulations, which are obtained by projecting the lower boundary part of a convex polytope in 3-space; see e.g. Edelsbrunner and Shah [28]. Regular triangulations are, thus, obviously optimal in the sense that they allow for a convex lifting surface. Some more optimality properties of \( DT(S) \) can be derived from this fact; see Musin [60]. Delaunay and regular triangulations can be constrained to live in non-convex polygons (rather than in the convex hull of the underlying point set \( S \)), see [48] and [1], respectively. Equiangularity and, with it, various other optimality properties carry over to constrained Delaunay triangulations [48].

\( DT(S) \) is the geometric dual of the famous Voronoi diagram of a point set \( S \) and can be computed in \( O(n \log n) \) time and \( O(n) \) space by various different approaches; see e.g. [8], [27], [34]. Su and Drysdale [71] gave a thorough experimental comparison of available Delaunay triangulation algorithms.

On the negative side, \( DT(S) \) fails to fulfill optimization criteria similar to those mentioned above, such as minimizing the maximum angle, or minimizing the longest edge. Edelsbrunner et al. [30], [29] gave \( O(n^2 \log n) \) time and \( O(n^2) \) time algorithms, respectively, for computing triangulations optimal in these respects. The former algorithm is based on an edge insertion paradigm which is shown in Bern et al. [10] to lead – in polynomial time – to triangulations with maximin triangle height, minmax triangle eccentricity, and minmax gradient surface, respectively.

**Minimum weight triangulations.** Most long-standing open was another optimal triangulation problem, the minimum weight triangulation. Here the criterion is weight, which is defined as the sum of all edge lengths. The complexity of computing a minimum weight triangulation, \( MWT(S) \), for arbitrary planar point sets \( S \) was open since 1975 when it was mentioned in Shamos and Hoey [67]. Minimum weight triangulation is included in Garey and Johnson’s [35] list of problems neither known to be NP-hard, nor known to be solvable in polynomial time. Very recently, its complexity status has been resolved; Mulzer and Rote [59] proved that minimum weight triangulation is NP-hard.

Earlier attempts to prove the minimum weight triangulation problem NP-hard have resulted in some related NP-completeness results; they are listed in [38]. Several heuristic algorithms have been proposed to solve this problem; see Lingas [53], Plaisted and Hong [63], and Heath and Pemmaraju [38]. None of these is known to produce a constant approximation in weight, although progress in this respect has been made later, see below.

It is well known that the Delaunay triangulation \( DT(S) \) may exceed \( MWT(S) \) in weight by a factor of \( \Theta(n) \); see Kirkpatrick [42]. Another popular triangulation, the greedy triangulation, \( GT(S) \), also may fail to approximate \( MWT(S) \) well. \( GT(S) \) is obtained by a greedy algorithm intended to yield small weight: Edges are inserted in increasing length order unless previously inserted edges are crossed and until the triangulation is completed. Several fast implementations have been proposed, e.g. by Dickerson et al. [22] who also give a brief history. Levcopoulos [49] showed a lower bound of \( \Omega(\sqrt{n}) \) on the approximation factor. A matching upper bound has been given in Levcopoulos and
Kriznaric [50]. The same paper introduces an interesting modification of GT(S) that achieves a constant weight approximation for MWT(S) in polynomial time, though with a very large constant. Very recently, Remy and Steger [65] developed an algorithm that finds an \((1 + \varepsilon)-\)approximation of MWT(S) in quasi-polynomial time, \(n^{O((\log n)\log n)}\).

For uniformly distributed point sets \(S\), both triangulations GT(S) and DT(S) yield satisfactory approximations for MWT(S); see e.g. Lingas [54]. In fact, GT(S) seems to perform slightly better, as reported in Dickerson et al. [23]. GT(S) can be constructed in \(O(n)\) expected time in this case, by an algorithm in Drysdale et al. [26] or by a modification of the algorithm in Levcopoulos and Lingas [52].

The weight of a triangulation may decrease when additional points (so-called Steiner points) are admitted. Eppstein [32] showed that the weight of a minimum weight Steiner triangulation for \(S\) may be \(\Omega(n)\) times smaller than the weight of MWT(S). The same paper gives efficient triangulation algorithms that approximate the weight of the former within a constant factor, thus improving over previous results in Bern et al. [12]. No polynomial-time algorithms are known for the exact minimum weight Steiner triangulation problem.

Dynamic programming is a powerful tool to deal with discrete optimization problems which are decomposable in a certain sense. It leads to polynomial-time solutions for some restricted instances of the minimum weight triangulation problem. For example, if \(S\) is the vertex set of a convex polygon then MWT(S) can be computed in \(O(n^3)\) time and \(O(n^2)\) space. The basic observation used is that – once some triangle of the triangulation has been fixed – the problem splits into subproblems (subpolygons) whose solutions can be found recursively, thereby avoiding recomputation of common subproblems. The triangulation method, first proposed by Gilbert [36] and Klincsek [44], does not really exploit convexity. It works as well for nonconvex polygons, and in fact for any interior face of a planar straight line graph. It is worth mentioning that, in the convex polygon case, MWT(S) is approximated by GT(S) up to a constant factor; see Levcopoulos and Lingas [51]. Anagnostou and Corneil [7] consider the case where \(S\) gives rise to a small number \(k\) of convex layers (nested convex hulls). Their dynamic programming approach works in time \(O(n^{3k+1})\) and thus is polynomial for constant \(k\). Meijer and Rappaport [58] later improved the bound to \(O(n^k)\) when \(S\) is restricted to lie on \(k\) non-intersecting line segments.

The minimum weight triangulation problem can also be formulated as a linear programming problem. To this end, a variable \(x_i\) is assigned to each of the \(\binom{n}{2}\) edges \(e_i\) defined by \(S\). The objective is to minimize

\[
\sum_{e_i} x_i |e_i|
\]

subject to the constraints

\[
0 \leq x_i \leq 1
\]
\[
x_i + x_j \leq 1 \text{ for } e_i \cap e_j \neq \emptyset
\]
\[
x_i + \sum_{e_j \cap e_i \neq \emptyset} x_j \geq 1
\]

The last two constraints express the property that a triangulation is a maximal set of non-crossing edges. An integer solution of this linear program yields a minimum weight triangulation: For each edge \(e_i\), inclusion into, or exclusion from MWT(S) is indicated by \(x_i = 1\) or \(x_i = 0\), respectively. An optimal solution need not be integer, however, and insisting on an integer solution results in an integer programming problem whose general version is known to be NP-complete [35]. Using a modified version of this approach, Ono et al. [62] were able to compute MWT(S) for up to a hundred points.

The afore-mentioned polynomial-time results for triangulating polygonal domains (rather than point sets) give motivation for the following

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**Related terms:**
- Dynamic programming
- Linear programming
- Minimum weight Steiner triangulation
- Subgraph approach
subgraph approach to compute \( MWT(S) \), proposed e.g. in Xu [73] and Cheng et al. [14]: Find a (suitable) subgraph \( G \) of \( MWT(S) \). If \( G \) contains \( k \) connected components, try all possibilities to add \( k - 1 \) edges to make it a connected graph \( C \). Complete each of these graphs \( C \) to a triangulation by optimally triangulating its faces, and select a triangulation with minimum weight, which gives \( MWT(S) \). This approach, which basically is exhaustive search, can be implemented to run in \( O(n^{k+2}) \) time. The problem, of course, is to find candidates for \( G \) with \( k \) small.

The subgraph approach should be distinguished from the heuristic approaches in [53, 63, 38] we mentioned before, which also first fix some graph \( G \) for \( S \) (for example, the minimum spanning tree) and then triangulate out its faces. The difference is that \( G \) will not be a subgraph of \( MWT(S) \) in general, and thus does not lead to an exact solution.

**Locally defined subgraphs of MWT.** Many efforts have been put into the study of subgraphs of \( MWT(S) \). Gilbert [36] pointed out that the shortest edge defined by \( S \) always belongs to \( MWT(S) \). Another simple observation is that unavoidable edges, which are edges not being crossed by any other edge in \( S \times S \), have to appear in any triangulation of \( S \) and thus are in \( MWT(S) \). For example, all edges of the convex hull of \( S \) are unavoidable. The number of unavoidable edges does not exceed \( 2n - 2 \), see Xu [74], and usually is very small.

Only in recent years, several more substantial subgraphs of \( MWT(S) \) have been identified. One of them arises from a class of empty neighborhood graphs introduced by Kirkpatrick and Radke [43]. Let \( p, q \in S \) and \( \beta \geq 1 \). The edge \( \overline{pq} \) is included in the \( \beta \)-skeleton, \( \beta(S) \), if the two discs of diameter \( \beta |pq| \) and passing through both \( p \) and \( q \) are empty of points in \( S \). It is not hard to see that \( \beta(S) \) always is a subgraph of the Delaunay triangulation \( DT(S) \). In fact, \( \beta(S) \) can be constructed from \( DT(S) \) in \( O(n) \) time; see Lingas [55] or Jaromczyk et al. [40].

Interestingly, \( \beta(S) \) is a subgraph of \( MWT(S) \) for \( \beta \) large enough. The original bound \( \beta \geq \sqrt{2} \) in Keil [41] has been improved later in Cheng and Xu [17] to \( \beta > 1.1768 \), which is close to the largest value \( 2/\sqrt{3} \) for which a counterexample is available [41]. Only for point sets \( S \) in convex position it is known that \( \beta > 2/\sqrt{3} \) always implies inclusion of \( \beta(S) \) in \( MWT(S) \); see Hainz et al. [37]. Whereas \( \beta(S) \) is a connected graph for \( \beta = 1 \) (known as the Gabriel graph of \( S \)), it may be highly disconnected for larger \( \beta \)-values. Cheng et al. [14] prove that – for uniformly distributed point sets \( S \) – the number of components is expected to be \( \Theta(n) \).

Yang et al. [76] formulated and proved a different inclusion region: If the union of the two disks with radius \( |pq| \) and centered at \( p \) and \( q \), respectively, is empty of points in \( S \), then \( \overline{pq} \) is an edge of \( MWT(S) \). That is, points \( p \) and \( q \) are mutual nearest neighbors. The skeleton generated in this way and the \( \beta \)-skeleton do not contain each other for \( \beta > 2/\sqrt{3} \), but for smaller \( \beta \)-values the \( \beta \)-skeleton contains the former as a subgraph. Note that both graphs are defined via a symmetric and local condition. A sufficient asymmetric condition can be found in Wang et al. [72]. We refer to Eppstein [33] for a survey paper on geometric graphs.

A distinct attempt to find a sufficient local condition defines an edge \( e \) as a light edge [2] if there is no edge in \( S \times S \) crossing \( e \) and shorter than \( e \). Let \( L(S) \) denote the graph formed by the light edges for \( S \). \( L(S) \) obviously contains all unavoidable edges, and in fact contains both Cheng and Xu’s 1.1768-skeleton and the skeleton in Yang et al. By construction, \( L(S) \) is a subgraph of the greedy triangulation \( GT(S) \), as light edges cannot be blocked by any edge previously inserted by the greedy strategy. On the other hand, \( L(S) \) is no subgraph of \( MWT(S) \) in general. However, if \( L(S) \) happens to be a full triangulation of \( S \), then \( L(S) = MWT(S) = GT(S) \). This allows for a fast computation of \( MWT(S) \) for a certain class of point sets \( S \), using greedy triangulation algorithms.
These results are observed in Aichholzer et al. [2] as a consequence of the following matching theorem for planar triangulations, proved in Aichholzer et al. [4] and in Cheng and Xu [16]: For any two triangulations $T_1$ and $T_2$ of a fixed point set $S$, there is a perfect matching between the edge set of $T_1$ and the edge set of $T_2$ such that matched edges either cross or are identical. A similar matching exists for the triangle intersection of the greedy triangulation $GT(S)$ into levels $L_1, \ldots, L_k$: Let $L_1 = L(S)$ be the set of edges that are light with respect to $S \times S$, let $L_2$ be the set of edges that are light with respect to $(S \times S) \setminus C_1$, where $C_1$ is the set of edges crossing $L_1$, and so on. Upper bounds on the ratio in weight between $GT(S)$ and $MWT(S)$ that depend on the number $k$ of levels are observed in Levcopoulos and Krznaric [50] and Aichholzer et al. [5]. In particular, $GT(S)$ is a constant-factor approximation of $MWT(S)$ provided $k = O(1)$.

**Globally defined subgraphs of MWT.** We have seen that several subgraphs of $MWT(S)$ can be found from local conditions, namely, via emptiness of particular inclusion regions. The subgraphs below are defined in a global way, via intersection of triangulations.

Call a triangulation $T$ of $S$ **locally minimal** if every 4-sided and point-empty polygon drawn by $T$ is optimally triangulated. That is, every convex quadrilateral contains the shorter one among its two diagonals. It is easy to see that $GT(S)$ is locally minimal and $DT(S)$, in general, is not. Let $LMT(S)$ denote the intersection of all locally minimal triangulations for $S$. Then $LMT(S)$ is a subgraph of $MWT(S)$, as this triangulation of course is locally minimal, too.

Whereas it is not known how to compute $LMT(S)$ in polynomial time, a surprisingly large subgraph of $LMT(S)$, the so-called **LMT-skeleton** can be identified by a simple and efficient method, proposed in Belleville et al. [9] and in Dickerson and Montague [24]: Consider some edge set $E \subset S \times S$. An edge $e \in E$ is called **redundant in $E$** if there is no convex quadrilateral formed by $E$ that has $e$ as its shorter diagonal. Edge $e$ is called **unavoidable in $E$** if no other edge in $E$ crosses $e$. The LMT-skeleton algorithms puts $E = S \times S$ and proceeds in several rounds. Each round first identifies all edges redundant in $E$ and eliminates them from the set, and then includes all edges that are unavoidable in the reduced set $E$ into the LMT-skeleton. The algorithm stops when no more edge in $E$ can be classified as either redundant or unavoidable. The number of rounds (but not the produced LMT-skeleton) depends on the ordering in which the edges are examined. In particular, there always exists an ordering such that one round suffices; see Hainz et al. [37].

The fact that the LMT-skeleton for $S$, and thus $LMT(S)$, tend to be connected even for large point sets comes at a surprise. From the practical point of view, the LMT-skeleton almost always nearly triangulates $S$. On the other hand, a 19-point counterexample to connectedness exists [9], and for uniformly distributed points, the expected number of components is $\Theta(n)$; see Bose et al. [13]. The constant of proportionality is extremely small, however. It is interesting to note that the LMT-skeleton, and the graph of light edges $L(S)$, exhibit a similar behavior of connectedness, but do not contain each other in general. We mention further that the improved LMT-algorithm in [37], that tends to yield some additional edges of $LMT(S)$, indeed constructs $LMT(S)$ provided the original LMT-skeleton for $S$ is connected.

lower weight bounds
locally minimal
LMT-skeleton
The LMT-skeleton clearly can be constructed in polynomial time, and several variants have been considered in order to gain efficiency [9], [24], [15], [37]. A powerful tool is pre-exclusion of edges before starting the LMT-algorithm, using the exclusion region in Das and Joseph [18]: For an edge e, consider the two triangular regions with base e and base angles $\frac{\pi}{3}$. If both regions contain points in $S$ then e cannot be part of $MWT(S)$. If $S$ is drawn from a uniform distribution, reduction to an expected linear number of candidate edges for $MWT(S)$ is achieved [22], and near-linear expected-time implementations of the LMT-algorithm exist [37], [69]. The LMT-skeleton based subgraph approach enables the computation of a minimum weight triangulation for some thousand points in reasonable time.

Some related and open problems. Let us briefly state a few open problems related to optimal triangulations.

A fast algorithm for computing the minimum weight triangulation of a simple polygon would have many applications and thus is of practical interest. Even for convex polygons, no algorithm using less than $\Theta(n^2)$ time and $\Theta(n^2)$ space is known [44]. No progress has been made since 1980 on this problem.

Minimality in weight may be relaxed to $k$-optimality, meaning that all $k$-sided and point-empty polygons in a triangulation are optimally triangulated. This is a generalization of local minimality which constitutes the instance $k = 4$. Whereas it is easy to compute 4-optimal triangulations (the greedy triangulation is one), no results are known on how to compute a 5-optimal triangulation in polynomial time. An algorithm based on the edge insertion paradigm is proposed in [75].

The maximal number of triangulations of a set of $n$ points is a quantity still not well understood. The gap between the best known upper bound, $O(43^n)$ [68], and lower bound, $\Omega(\sqrt{72}^n)$ [6], is large. The common belief is that the latter function is closer to the truth.

In the last ten years, a relaxation of triangulations, so-called pseudo-triangulations, have been become popular, especially in computational geometry. In addition to triangles, pseudo-triangles (polygons with exactly three convex internal angles) are used as faces. Unfortunately, not much is known about optimality properties of pseudo-triangulations. Some basic properties of minimum weight pseudo-triangulations are given in the paper [3], which also shows that the greedy pseudo-triangulation always has to be a triangulation.

References


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