Monotone and near-monotone networks

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Investigate systems using robust, structural information!
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1. Monotone systems: definitions and basic results
preliminaries

given dyn. system: \[ \frac{d}{dt} x = f(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \]

\(\preceq\) and is an orthant order on \(\mathbb{R}^n\) to \((\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n\):

- \(x \preceq y \iff \forall i : \epsilon_i(y_i - x_i) \geq 0\)
- \(x \prec y \iff \forall i : \epsilon_i(y_i - x_i) \geq 0\) and \(x \neq y\)
- \(x \ll y \iff \forall i : \epsilon_i(y_i - x_i) > 0\)
species graphs

species graph \( G = (V, E, L) \) to \( f : \Leftrightarrow \)

- vertices \( V = \{x_1, \ldots, x_n\} \)
- edges \( \forall i \neq j : e_{ij} \in E \Leftrightarrow \frac{\partial f_j}{\partial x_i} \neq 0 \) \hspace{1cm} \text{influence of } x_i \text{ to } x_j
- labels \( L : E \to \{-1, 0, 1\} \) \hspace{1cm} \text{sign of influence}

\[
L(e_{ij}) = l_{ij} =: \begin{cases}  
+1 & Df_{ji} = \frac{\partial f_i}{\partial x_j} \geq 0 \\
-1 & Df_{ji} = \frac{\partial f_i}{\partial x_j} \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]
\[
\begin{align*}
\dot{x}_1 &= 1 + x_2^3 \\
\dot{x}_2 &= -x_2 + x_4 \\
\dot{x}_3 &= x_1^3 + \tanh(x_2) \\
\dot{x}_4 &= \exp(-x_1) - x_3
\end{align*}
\]
spin assignment

spin assignment $\sigma \iff \sigma : V \rightarrow \{-1, +1\}$

$\sigma$ consistent with $e_{ij} \in E \iff l_{ij}\sigma_i\sigma_j = 1$

$\sigma$ is consistent with $G \iff \forall e_{ij} \in E : l_{ij}\sigma_i\sigma_j = 1$

**Theorem**

There is a consistent spin assignment for $G$ iff every undirected loop in $G$ contains an even number of negative labels and no zero labels.

If this holds $G$ is called a balanced graph.
example

not balanced

balanced
example

not balanced

balanced

+ + + +

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\[ \dot{x} = f(x) \quad (\ast) \]

- let \( x(t) \) and \( z(t) \) be solutions to (\ast)
- perturbation in initial condition: \( x(0) \preceq z(0) \)
- systems with balanced graphs: “coherent” reaction to perturbations

\[ f \] defines a monotone system via (\ast) to the order \( \preceq \) :\( \iff \forall t > 0 \]
\[ x(t) \preceq z(t) \]
\[
\dot{x} = f(x) \quad (*)
\]

- let \(x(t)\) and \(z(t)\) be solutions to \((*)\)
- perturbation in initial condition: \(x(0) \prec z(0)\)
- systems with balanced graphs: “coherent” reaction to perturbations

\[f\] defines a strongly monotone system via \((*)\) to the order \(\prec\):
\[
\iff \forall t > 0 \; x(t) \ll z(t)
\]
Kamke’s theorem

Let $G$ be the species graph of $f$.

**Theorem**

*If $G$ is balanced, then $f$ defines a monotone system (to the order induced by the spin assignment).*
G is strongly connected :⇔ ∀ i ≠ j : there is a directed path from x_i to x_j

f defines a strongly monotone dyn. system ⇐ G is balanced, strongly connect and all non-zero components of Df are everywhere non-zero

\[
\begin{align*}
\dot{x}_1 &= 1 + x_2^3 \\
\dot{x}_2 &= -x_2 + x_4 \\
\dot{x}_3 &= x_1^3 \\
\dot{x}_4 &= \exp(-x_1) - x_3
\end{align*}
\]
Hirsch Generic Convergence theorem

Let $f$ define a strongly monotone system and let all solution to
\[ \dot{x} = f(x) \] (\text{*}) be bounded.

**Theorem**

*All generic solutions to (\text{*}) converge to the set of steady-states.*

$\Rightarrow$ No oscillations, no chaotic behavior!
2. Monotone input-output systems (MIOS)
Decomposition of non-monotone systems to interacting MIOS!
motivation

Decomposition of non-monotone systems to interacting MIOS!
Decomposition of complex systems to interacting MIOS and aMIOS
I/O system:

- \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \)
- \( \dot{x} = f(x, u), \ y = h(x) \) (\( \ast \))
- assume orders in \( \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p \) all denoted by \( \prec \)
- \( \phi(t, x, u) \) denotes the solution to (\( \ast \)) at time \( t \), initial cond. \( x \) and input \( u \)

(\( \ast \)) is a monotone I/O system (MIOS) \( \iff \)

- \( \forall x_1 \prec x_2 \) and \( u_1(t) \prec u_2(t) \) \( \Rightarrow \) \( \phi(t, x_1, u_1) \prec \phi(t, x_2, u_2) \)
- \( h \) is a monotone map (preserves \( \prec \)), i.e.
  \( x_1 \prec x_2 \) \( \Rightarrow \) \( h(x_1) \prec h(x_2) \)
generalized species graph

\[
\dot{x} = f(x, u), \quad y = h(x) \quad (*)
\]

\[x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad y(t) \in \mathbb{R}^p\]

species graph for I/Os (\(\ast\)):

- \(V = \{x_1, \ldots, x_n, u_1, \ldots, u_m, y_1, \ldots, y_p\}\)
- additional edges corresponding to \(\frac{\partial f_i}{\partial u_j}, \frac{\partial h_i}{\partial x_j}\)

Theorem (generalized Kamke’s theorem)

If the species graph of (\(\ast\)) is balanced (+“some technical assumptions”), then (\(\ast\)) is a MIOS.
consider: $\dot{x} = f(x, u), u(t) \in \mathbb{R}, y(t) \in \mathbb{R}$ \hspace{1cm} (\star)

(\star) has a unique characteristic \iff for constant input $u(t) = u_0$

solutions of (\star) converge to unique stable state $K(u_0)$

characteristic $k$ of $f$ is defined via:

$$k(u_0) = h(K(u_0))$$
two MIOS = positive feedback

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, u_1), \quad y_1 = h_1(x_1), \quad \text{char. } k, \quad u_1 = y_2 \\
\dot{x}_2 &= f_2(x_2, u_2), \quad y_2 = h_2(x_2), \quad \text{char. } g, \quad u_2 = y_1
\end{align*}
\]

Intuition: consider memoryless iterated system

\[
\begin{align*}
    u_2^n &= k(u_1^n) \\
    u_1^{n+1} &= g(u_2^n)
\end{align*}
\]

fixed points: \( k(u_2) = g^{-1}(u_2) \)
stable: \( k'(u_2) < (g^{-1})'(u_2) \)
\[ S \iff k' < (g^{-1})' \quad \rightarrow \text{stable!?!} \]

\[ U \iff k' > (g^{-1})' \quad \rightarrow \text{unstable!?!} \]
basic theorem: positive feedback

Theorem
Consider the situation described above. All critical points labeled “S” are asymptotically stable and all critical points labeled “U” are unstable.

remarks:
• method is generalization of phase plan analysis for 2d systems
• the two MIOS can be of arbitrary dimension
• stability analysis only using information about topology and characteristics
extension: negative feedback

\[
\dot{x} = f(x, u), \ y = h(x) \quad (*)
\]

\( (*) \) is anti-monotone I/O system (aMIOS) \( \iff \)

- \( \forall x_1 \preceq x_2 \text{ and } u_1(t) \preceq u_2(t) \implies \phi(t, x_1, u_1) \preceq \phi(t, x_2, u_2) \)
- \( h \) reverses \( \preceq \): \( \forall x_1 \preceq x_2 \to h(x_2) \preceq h(x_1) \)
basic result: negative feedback

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, u_1), \quad y_1 = h_1(x_1), \quad \text{MIOS: char. } k, \quad u_1 = y_2 \quad (*) \\
\dot{x}_2 &= f_2(x_2, u_2), \quad y_2 = h_2(x_2), \quad \text{aMIOS: char. } g, \quad u_2 = y_1 \quad (*)
\end{align*}
\]

consider the iterated system: \[u^{n+1} = (g \circ k)(u^n) \quad (**)

**Theorem**  
If the iterated system (**) has a globally attractive fixed point, then the feedback system (*) has a globally attractive steady state.
example: testosterone model

\[ \dot{x}_1 = \frac{a_1}{a_2 + x_3} - a_3 x_1 \]
\[ \dot{x}_2 = b_1 x_1 - b_2 x_2 \]
\[ \dot{x}_3 = c_1 x_2 - c_2 x_3 \]

decomposition

MIOS

\[ \dot{x}_1 = u - a_3 x_1 \]
\[ \dot{x}_2 = b_1 x_1 - b_2 x_2 \]
\[ h_1(x_1, x_2) = c_1 x_2 \]
\[ k(u) = \frac{b_1 c_1}{b_2 a_3} u \]

aMIOS

\[ \dot{x}_3 = u - c_2 x_3 \]
\[ h_2(x_3) = \frac{a_1}{a_2 + x_3} \]
\[ g(u) = \frac{a_1 c_2}{c_2 a_2 + u} \]
example: testosterone model

iterated map: \[(g \circ k)(u) = \frac{p}{q+u}\]

⇒ unique steady state!
3. Summary
summary

- monotone systems can be identified via their species graph
- only little and robust information (labeled graph) is needed
- convergence of strongly monotone systems to steady-states
- decomposition of complex systems to simpler MIOS
- identification of steady-states via characteristics