Balancing at the border of instability

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presented by Helmut Hauser
Overview

1) observations in natural neural systems
2) two examples
3) older approaches
4) new approach
5) proof of the new concept
Some observations

In biological neural circuits:

e.g. Eye movements: saccadic movements, in between the eye keeps still due a constant level of neural activity.

But neural activity has a natural tendency to decay!

→ a question raises: How can cause a transient stimulus persistent changes in neural activity?
Some observations

according to a long-standing hypothesis, persistent neural activity is maintained by synaptic feedback loops. (positive feedback works against the natural decay !)

feedback too weak  →  decay
feedback too strong  →  bifurcation  →  leads to instability

IDEAL: working at the “border of instability”
(have to balance exactly the decay)  →  fine tuning

Note: biological systems are very robust !!
Some approaches

Older approaches to model fine tuning:

- gradient descent and function approximation algorithms [Arnold et al],[Seung et al]

- feedback learning on differential anti-Hebbian synaptic plasticity [X.Xie and H.S.Seung]

→ still remains unclear how the required fine tuning is physiological feasible.
Some approaches

differential model for neuronal integration based on bistability has recently been proposed. [A. Koulakov et al]

Sonntag’s and Moreau’s approach

Hypothesis of precisely tuned synaptic feedback. Propose an adaptation mechanism for fine tuning of a neural integrator

may explain the experimentally observed robustness of neural integrators with respect to perturbations.
2 biological examples

1.) persistent neural activity in oculomotor control system

- natural tendency to decay with relaxation time 5-100 ms
- positive synaptic feedback works against it
- again if feedback is too strong $\rightarrow$ bifurcations $\rightarrow$ unstable
- so-called neural integrator is used to maintain persistent neural activity
2) hair cells in the cochlea (auditory system)

- hair cells operate as nanosensors, which transform acoustic stimuli into electric signals.
- almost self-oscillating system
- low concentrations of \( \text{Ca}^{2+} \) ➔ oscillations are damped by the viscous fluid
- high \( \text{Ca}^{2+} \) concentrations ➔ system undergoes a Hopf bifurcation
- working at the border ➔ even a weak stimulus can cause a detectable oscillation
Approach

Designing a neural integrator with self-tuning feedback

- State variable $x$ is the neural activity
- $u(t)$ is input from presynaptic neurons
- $\mu$ is the bifurcation parameter ($\mu_0 =$ critical value)
- adaptation law shouldn‘t depend on $\mu_0$
- if $x=0$, it stays 0 $\rightarrow$ strictly positive values for $x$
- adaptation law may depend on $x$ and $\mu$
Approach

Designing a neural integrator with self-tuning feedback

\[ \dot{x} = \mu x - x \mu_0 + u(t) \]

\[ \dot{\mu} = f(x) - g(\mu) \]
Approach

Designing a neural integrator with self-tuning feedback

\[ \dot{x} = \mu \cdot x - x \cdot \mu_0 \]

\[ \dot{\mu} = f(x) - g(\mu) \quad \text{adaptation law} \]

\[- \mu_0 \cdot x \quad \ldots \text{natural decay}\]

\[ \mu \cdot x \quad \ldots \text{positive synaptic feedback} \]
Approach

\[ \dot{x} = \mu \cdot x - x \cdot \mu_0 \]
\[ \dot{\mu} = f(x) - g(\mu) \]

3 conditions:

(1) \( g(\mu) \) has to be a strictly increasing function \( \Rightarrow -g(\mu) \) is then negative feedback

(2) There exist a \( x^* \) such that \( f(x^*) = g(\mu_0) \) \( \Rightarrow \) if neural activity is constant \( x^* \) \( \Rightarrow \) \( \mu \) would relax to \( \mu_0 \)

(3) \( f(x) \) has to be a strictly decreasing function \( \Rightarrow \) level of neural activity negatively regulates synaptic feedback strength
Let $\mu_0 \in \mathbb{R}$ and consider continuously differentiable functions $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$.

Assume that $f$ is strictly decreasing, $g$ is strictly increasing, and $g(\mu_0)$ is in the image of $f$. Then the nonlinear system (1) and (2) with $x \in \mathbb{R}_{>0}$ and $\mu \in \mathbb{R}$ has a unique equilibrium point $(f^{-1}(g(\mu_0)), \mu_0)$, which is globally asymptotically stable.

\[
\begin{align*}
\dot{x} &= \mu \cdot x - x \cdot \mu_0 \quad (1) \\
\dot{\mu} &= f(x) - g(\mu) \quad (2)
\end{align*}
\]
Barbashin theorem

Let \( x = 0 \) be an equilibrium point for \( \dot{x} = f(x) \)

Let \( V: D \rightarrow \mathbb{R} \) be a continuously differentiable positive definite function on a domain \( D \) containing the origin \( x=0 \), such that \( \dot{V} \leq 0 \) in \( D \).

Let \( S = \{ x \in D \mid \dot{V}(x) = 0 \} \) and suppose that no solution stays identically in \( S \), other than the trivial solution \( x(t)=0 \).

Then the origin is asymptotically stable.
FIG. 1. Tuning of a neural integrator. Simulation of Eq. (1) with $\dot{\mu} = \xi(-ax - b\mu + c)$ and $u(t) = \sum_i \delta(t - t_i)\{x_{\text{desired}}(t_i+) - x(t_i)\}$, where $\delta(\cdot)$ represents the Dirac impulse and where the sum goes over all saccade times $t_i$. The constants are $\mu_0 = 200 \text{ s}^{-1}$, $a = 1 \text{ s}^{-1}$, $b = 0.01 \text{ s}^{-1}$, $c = 42 \text{ s}^{-2}$, and $\xi = 0.01$. The adaptation law satisfies the compatibility condition $a(20 + 60 \text{ Hz})/2 + b\mu_0 = c$. 

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FIG. 3. Tuning of a nonlinear oscillator. Simulation of equations $\ddot{x} + (\mu_0 - \mu)\dot{x} + \lambda \dot{x}^3 + \omega^2 x = 0$ and $\dot{\mu} = f(r) - g(\mu)$ with $\mu_0 = 1$, $\lambda = 1$, $\omega = 1$ and $f(r) - g(\mu) = 1/(1 + r^2) - \mu + 1/2$. The variable $r$ is determined by $r^2 = x^2 + (\dot{x}/\omega)^2$. 

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Hassan K. Khalil,