A COUNTABLE BASIS FOR $\Sigma^1_2$ SETS
AND RECURSION THEORY ON $\aleph_1$

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ABSTRACT. Countably many $\aleph_1$-recursively enumerable sets are constructed from which all the $\aleph_1$-recursively enumerable sets can be generated by using countable union and countable intersection. This implies under $V = L$ that there exists as well a countable basis for $\Sigma^1_n$ sets of reals, $n > 2$. Further under $V = L$ the lattice $\mathcal{E}^*(\aleph_1)$ of $\aleph_1$-recursively enumerable sets modulo countable sets has only $\aleph_1$ many automorphisms.

Let $\mathcal{E}$ denote the lattice of recursively enumerable (r.e.) sets under inclusion, and let $\mathcal{E}^*$ denote the quotient lattice of $\mathcal{E}$ modulo the ideal of finite sets. Both structures have been extensively studied (see e.g. the survey by Soare [5]). In recent years research has concentrated on the existence of automorphisms and the decidability of the elementary theory.

Analogous questions arise in $\alpha$-recursion theory for admissible ordinals $\alpha$. Here one studies the lattice $\mathcal{E}(\alpha)$ of $\alpha$-r.e. subsets of $\alpha$ and the quotient lattice $\mathcal{E}^*(\alpha)$ modulo the ideal of $\alpha^*$-finite sets (see e.g. the survey by Lerman [2]). A set is $\alpha$-r.e. iff it is definable over $L_\alpha$ by some $\Sigma_1$ formula with parameters. A function is $\alpha$-recursive iff its graph is $\alpha$-r.e. A set is $\alpha^*$-finite iff every $\alpha$-r.e. subset of it is $\alpha$-recursive. For simplicity we assume $V = L$ in the first part of this paper where we study $\aleph_1$-r.e. sets.

Lachlan has proved the following basic result about automorphisms of $\mathcal{E}^*$ (see Soare [4]): There are $2^{\aleph_0}$ automorphisms of $\mathcal{E}^*$. Sutner [7] has noticed that one can use Lachlan's construction in order to show that for all countable admissible $\alpha$ there are $2^\alpha$ many automorphisms of $\mathcal{E}^*(\alpha)$. The argument breaks down for $\alpha = \aleph_1$ despite the fact that $\aleph_1$ is like $\omega$ a regular cardinal. Observe that in the case $\alpha = \aleph_1$ the $\alpha^*$-finite sets are just the countable sets. We show in this paper that there are in fact only $\aleph_1$ (instead of $2^{\aleph_1}$) many automorphisms of $\mathcal{E}^*(\aleph_1)$.

DEFINITION 1. We say that a class $\Gamma$ of sets has a countable basis ($A_n)_{n \in \omega}$ if $\{A_n|n \in \omega\} \subseteq \Gamma$ and $\Gamma$ is the closure of $\{A_n|n \in \omega\}$ under countable unions and intersections.

Observe that the class of $\aleph_1$-r.e. sets is closed under countable unions and intersections.

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THEOREM 2. The class of \( \mathbb{N}_1 \)-r.e. sets has a countable basis \( (A_n)_{n \in \omega} \). In fact every \( \mathbb{N}_1 \)-r.e. set can be written as a countable intersection of countable unions of countable intersections of the sets \( (A_n)_{n \in \omega} \).

PROOF. Take a universal \( \mathbb{N}_1 \)-r.e. set \( W \) such that \( (W_e)_{e \in \mathbb{N}_1} \) is an enumeration of all \( \mathbb{N}_1 \)-r.e. sets, where \( W_e := \{ \delta \langle e, \delta \rangle \in W \} \). Further take an \( \mathbb{N}_1 \)-recursive function \( C \) from \( \mathbb{N}_1 \) into \( \wp(\mathbb{N}) \) such that \( \{ C(e) | e \in \mathbb{N}_1 \} \) is a family of almost disjoint sets (i.e. every \( C(e) \) is infinite and \( C(e) \cap C(e') \) is finite for \( e \neq e' \), see e.g. Kunen [1]).

We construct first countably many \( \mathbb{N}_1 \)-r.e. sets \( (A_n)_{n \in \omega} \) such that for every \( e \in \mathbb{N}_1 \) with \( e \geq \omega \)

\[
W_e - e = \left( \bigcup_{j \in \omega} \left( \bigcap_{n \in \omega} \{ A_n | n \in C(e) \land n > j \} \right) \right) - e.
\]

The sets \( (A_n)_{n \in \omega} \) are constructed simultaneously in \( \mathbb{N}_1 \) many steps. At step \( \gamma \) we determine for every \( n \) on which fact it depends whether or not \( e \) is enumerated in \( A_n \).

We assign in an \( \mathbb{N}_1 \)-recursive way to every \( e \in \mathbb{N}_1 \) a function \( p_e \in L_n \) which maps \( \omega \) one-one onto \( \gamma + 1 \). For \( e < \gamma \) one might consider \( p_e^{-1}(e) \) as the priority of the equality \( W_e = \bigcup_{j < \omega} \left( \bigcap_{n \in \omega} \{ A_n | n \in C(e) \land n > j \} \right) \) at step \( \gamma \). We change priorities at every step because it is important that the priority list is never longer than \( \omega \).

Step \( \gamma \) (\( \omega < \gamma < \mathbb{N}_1 \)). For \( n \in C(p_{\gamma}(0)) \) we determine that \( \gamma \) is put in \( A_n \) if and only if \( \gamma \) is enumerated in \( W_{p_{\gamma}(0)} \). For \( j > 0 \) and \( n \in (C(p_{\gamma}(j)) - \bigcup_{j < \gamma} C(p_{\gamma}(j))) \) we determine that \( \gamma \) is put in \( A_n \) if and only if \( \gamma \) is enumerated in \( W_{p_{\gamma}(j)} \). For \( n \in \omega - \bigcup_{e \leq \gamma} C(e) \) it does not matter whether we put \( \gamma \) in \( A_n \) or not.

It is obvious from the construction that the sets \( A_n \) are \( \mathbb{N}_1 \)-r.e. Further for \( \omega < e < \gamma \) we have

\[
\gamma \in \bigcup_{j \in \omega} \left( \bigcap_{n \in \omega} \{ A_n | n \in C(e) \land n > j \} \right)
\]

\[
\Leftrightarrow \gamma \in \bigcap_{n \in \omega} \{ A_n | n \in C(e) \land n > \max \{ C(e) \cap \left( \bigcup \{ (C(e')) | p_{\gamma}(e') < p_{\gamma}(e) \} \} \} \}
\]

So far we cannot generate every set \( W_e \) with countable unions and intersections from the basis elements without making mistakes at countably many points. Therefore we add countably many further \( \mathbb{N}_1 \)-r.e. sets to the constructed basis elements \( (A_n)_{n \in \omega} \) which enable us to correct these mistakes. Let \( f \) be an \( \mathbb{N}_1 \)-recursive function which maps \( \mathbb{N}_1 \) one-one into \( \wp(\mathbb{N}) \). Define \( \mathbb{N}_1 \)-recursive sets \( (R_n)_{n \in \omega} \) by \( \gamma \in R_n : \Leftrightarrow n \in f(\gamma) \). We add then the sets \( (R_n)_{n \in \omega} \) and \( (\mathbb{N}_1 - R_n)_{n \in \omega} \) to the basis. For every \( \gamma \in \mathbb{N}_1 \) we have

\[
\{ \gamma \} = \bigcap_{n \in f(\gamma)} R_n \cap \bigcap_{n \notin f(\gamma)} (\mathbb{N}_1 - R_n).
\]
Thus we can write every countable set as a countable union of countable intersections and the complement of every countable set as a countable intersection of countable unions of basis elements. Therefore we can correct every mistake on countably many points.

**Corollary 3.** There are $\aleph_1$ automorphisms of $\mathcal{P}^*(\aleph_1)$.

**Proof.** It is obvious that one can construct $\aleph_1$ many $\aleph_1$-recursive permutations of $\aleph_1$ which induce different automorphisms of $\mathcal{P}^*(\aleph_1)$. On the other hand every automorphism $\Phi$ of $\mathcal{P}^*(\aleph_1)$ preserves countable unions and intersections. Therefore $\Phi$ is completely determined by the values $(\Phi(A_n))_{n\in\omega}$, where $(A_n)_{n\in\omega}$ is a basis for the $\aleph_1$-r.e. sets and $(A_\ast)_{n\in\omega}$ are the corresponding equivalence classes in $\mathcal{P}^*(\aleph_1)$.

We now leave $\alpha$-recursion theory and the assumption $V = L$ and turn to descriptive set theory in ZFC. It makes sense to ask whether the classes $\Sigma^1_n$ and $\Pi^1_n$ have a countable basis according to Definition 1 since these classes are closed under countable union and intersection. Obviously if $\Sigma^1_n$ has a countable basis then the complements of the basis elements form a basis for $\Pi^1_n$ and vice versa. Observe that $\Delta^1_1$, the class of Borel sets, has a countable basis. If one chooses suitable basis elements one can generate the Borel hierarchy without using complementation.

**Corollary 4.** Assume $n \geq 2$ and $\omega \subseteq L[a]$ for some $a \subseteq \omega$. Then $\Sigma^1_n$ has a countable basis.

**Proof.** It is well known that for every $m \geq 1$ a subset of $\omega$ is $\Sigma^1_{m+1}$ iff it is $\Sigma^1_m$ definable over HC. Under the assumption $\omega \subseteq L[a]$ we have HC = HC$^{L[a]} = L[a]$. Thus the $\Sigma^1_2$ sets are just the sets which are $\Sigma_1$ definable over $L[a]$ and for $m \geq 2$ the $\Sigma^1_{m+1}$ sets are just the sets which are $\Sigma_1$ definable over $\langle L_n[a], e, P_m \rangle$ with a suitable mastercode $P_m$. Since one can define a map which maps $\omega$ one-one onto $\aleph_1$, by a $\Delta_1$ definition over $L_n[a]$, it does not matter whether one considers subsets of $\omega_1$ or of $\aleph_1$. Further the construction of a countable basis in the proof of Theorem 2 works as well for $L_n[a]$ and $\langle L_n[a], e, P_m \rangle$ instead of $L_n$.

**Remark 5.** Richard Mansfield has shown [3] that any countably generated $\sigma$-algebra consisting entirely of Lebesgue measurable sets does not contain all $\Sigma^1_1$ sets. Therefore $\Sigma^1_1$ has no countable basis in the sense of Definition 1 if all $\Sigma^1_n$ sets are measurable. This implies that $\Sigma^1_1$ never has a countable basis. Further, Solovay’s model of ZFC where all projective sets are measurable [6] supplies an example where no $\Sigma^1_n$ has a countable basis.

In addition Mansfield has given a complete answer for $\Sigma^1_2$: If $\Sigma^1_2$ has a countable basis then $\omega \subseteq L[a]$ for some $a \subseteq \omega$ (to appear).

**References**