Annals of Pure and Applied Logic 24 (1983) 189–212
North-Holland

THE INTERVALS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS DETERMINED BY MAJOR SUBSETS*

Wolfgang MAASS
Department of Mathematics, University of Illinois, Chicago, IL 60650, USA

Michael STOB
Department of Mathematics, Calvin College, Grand Rapids, MI 49506, USA

Received 1 June 1981

Introduction

If X is any subset of the set N of natural numbers, let $\mathscr{E}(X)$ denote the lattice formed by the sets $\{W \cap X\colon W \text{ is r.e.}\}$ under inclusion. Let $\mathscr{E}^*(X)$ be the lattice $\mathscr{E}(X)$ modulo the ideal of finite sets. Let $\mathscr{E}$ ($\mathscr{E}^*$) abbreviate $\mathscr{E}(N)$ ($\mathscr{E}^*(N)$). Further if $A \in \mathscr{E}(X)$, let $A^* \in \mathscr{E}^*(X)$ denote the equivalence class of A and $A =^* B$ ($A \subseteq^* B$) denote $A^* = B^*$ ($A^* \subseteq B^*$). If $B, A \in \mathscr{E}$ such that $B \subseteq A$ and $A - B$ is infinite, then B is a major subset of A ($B \subset_{\mathrm{m}} A$) if $W \cup A =^* N$ implies $W \cup B =^* N$ for all r.e. sets W. Our main result is

Theorem. *If $B \subset_{\mathrm{m}} A$ and $\hat{B} \subset_{\mathrm{m}} \hat{A}$, then $\mathscr{E}^*(A - B) \cong \mathscr{E}^*(\hat{A} - \hat{B})$.*

The proof of this theorem is by a modification and considerable extension of the automorphism machinery of Soare [6]. The construction and proof occupy Sections 4 and 5.

Section 1 of this paper provides background information on major subsets and their importance for the general program of determining the structure of $\mathscr{E}$.

In addition, we introduce and prove a simple lemma, the Marker Lemma, which is useful in many constructions involving d.r.e. sets (sets which are the differences of r.e. sets). In Section 2 we give a new construction of major subsets which also provides more information on the various types of major subsets that can be constructed. (Although our theorem says that all major subsets B of A are alike in so far as $\mathscr{E}^*(A - B)$ is concerned, there are other properties which distinguish various types of major subsets.) In Section 3 we introduce the finite splitting property, a property which all major subsets have and which is crucial to our proof of the main theorem. The paper concludes in Section 6 with some corollaries, remarks, and open questions.

* During preparation of this paper the first author was supported by the Heisenberg Programm der Deutschen Forschungsgemeinschaft, West Germany. The second author was supported by NSF Grant MCS 80-02937.

1. Major subsets

Two interralated programs for studying the structure of $\mathscr{E}$ have been the classification of the elementary theory of $\mathscr{E}$ and the characterizations of Aut($\mathscr{E}$), the group of automorphisms of $\mathscr{E}$. In both programs, major subsets have played important technical roles. Lachlan [3] used major subsets in his decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$ to construct canonical embeddings of each finite distributive lattice into $\mathscr{E}^*$ which enabled him to give necessary and sufficient conditions for an $\forall\exists$-sentence about $\mathscr{E}^*$ to be true. Lachlan [2], and Lerman, Shore, and Soare [4] have used major subsets to answer questions about which lattices are $\mathscr{E}^*(\bar{A})$ for r.e. sets A and about whether the lattice $\mathscr{E}^*(\bar{A})$ determines the automorphism type of A.

It is this program of characterizing lattices of the form $\mathscr{E}^*(\bar{A})$ for which major subsets are most useful. Lachlan [2] gave a complete characterization of those Boolean algebras B such that $B \cong \mathscr{E}^*(\bar{A})$ for some r.e. A. However little is known about the remaining lattices. (Of course if there is a $\Phi \in \mathrm{Aut}(\mathscr{E})$ such that $\Phi(A) = \Phi(B)$, then $\mathscr{E}^*(\bar{A}) \cong \mathscr{E}^*(\bar{B})$ so that a classification of such lattices is important for the general program of characterizing Aut($\mathscr{E}$).) Lachlan used major subsets to give an example of a lattice of the form $\mathscr{E}^*(\bar{B})$ with no complemented elements. For let A be a coatom in $\mathscr{E}^*$ (i.e., A is a maximal r.e. set) and let $B \subset_m A$. Then $\mathscr{E}^*(\bar{B})$ has no complemented elements. The set B is called *r-maximal* for no recursive set R splits $\bar{B}$ nontrivially. A first corollary of our theorem is that each r-maximal set B which arises in this way as a major subset of a maximal set has the same $\mathscr{E}^*(\bar{B})$. For let $\mathscr{M}^* = \mathscr{E}^*(A - B)$ which is unique by the theorem. $\mathscr{E}^*(\bar{B})$ is formed from $\mathscr{M}^*$ by adding one new element which is greater than each element of $\mathscr{M}^*$. Lachlan [2] also constructed r-maximal sets which are not subsets of maximal sets. For such an r-maximal B, $B \subset_m A$ for every superset A of B that is coinfinite. Thus although our theorem does not lead to a classification of $\mathscr{E}^*(\bar{B})$ for such B, it does tell us that such a lattice has all its proper initial segments isomorphic to $\mathscr{M}^*$. Again, Lachlan used major subsets to give an example of lattice $\mathscr{E}^*(\bar{B})$, such that no interval of $\mathscr{E}^*(\bar{B})$ is a Boolean algebra. For let A be any nonrecursive set and $\hat{B} \subset_m A$. Let f be a 1–1 recursive function mapping N onto A. Then letting $B := f^{-1}(\hat{B})$, our theorem gives us that $\mathscr{E}^*(\bar{B}) \cong \mathscr{M}^*$. But $\mathscr{M}^*$ has all its intervals isomorphic to $\mathscr{M}^*$ for if $B \subseteq C \subseteq D \subseteq A$ and $B \subset_m A$, then $C \subset_m D$. But is is easy to show that $\mathscr{M}^*$ is not a Boolean algebra.

Because of the usefulness of major subsets in studying $\mathscr{E}^*(\bar{B})$, study of the lattices $\mathscr{E}^*(A - B)$ where $B \subset_m A$ was initiated in Herrmann [1] and Stob [9]. Numerous uniformities were discovered. For instance both Herrmann and Stob showed

Theorem 1.1. *Suppose $B \subset_m A$. Then there is an r.e. set C such that $B \subseteq C \subseteq A$, $C - B$ is infinite, and for every r.e. W, if $W \supseteq A - C$, then $W \supseteq C - B$.*

Notice that Theorem 1.1 implies that $C \cap (A-B)$ is not complemented in $\mathscr{E}^*(A-B)$ in a very strong way. Theorem 1.1 also gives an elementary difference between $\mathscr{M}^*$ and $\mathscr{E}^*$.

Using theorems such as 1.1 Stob [1980] showed

Theorem 1.2. *The $\forall\exists$-theories of the lattices $\mathscr{E}^*(A-B)$ for $B \subset_m A$ are all the same and this common theory is decidable.*

From such uniformities, Lerman and Herrmann were led to conjecture that $B \subset_m A$ determines $\mathscr{E}^*(A-B)$ which is what we show in the paper.

Our proof of the main theorem is an adaptation of the automorphism machinery first developed by Soare [6] to show that any two maximal elements of $\mathscr{E}^*$ were automorphic. This machinery has been extended in various ways by Maass [5], Soare [7] and Stob [8] to answer questions concerning the isomorphism types $\mathscr{E}^*(\bar{A})$ and concerning properties of r.e. sets which are invariant under $\mathrm{Aut}(\mathscr{E})$. Each of the extensions was developed to build an isomorphism of $\mathscr{E}^*(A-B)$ and $\mathscr{E}^*(\hat{A}-\hat{B})$ where A, B, $\hat{A}$ and $\hat{B}$ are r.e. sets. Our proof, besides incorporating all of the machinery of the original construction, requires two further major ingredients. The first is a new splitting property, the finite splitting property, which was introduced in a slightly weaker form by Maas [5]. This property is discussed in Section 2. The second ingredient is summarized in what we have called the Marker Lemma (Lemma 1.3) and was essentially introduced by Lachlan [2] in the so-called refinement theorem of his decision procedure. We informally describe that technique here using a simple example from the proof of our theorem.

Given $B \subset_m A$ and $\hat{B} \subset_m \hat{A}$ we must construct an isomorphism Φ from $\mathscr{E}^*(A-B)$ to $\mathscr{E}^*(\hat{A}-\hat{B})$. Suppose then that U is an r.e. set. We must then enumerate an r.e. set $\hat{U}$ so that $\hat{U}$ is the image of U under Φ. In particular we must meet the requirement

$$U \cap (A-B) \text{ is infinite iff } \hat{U} \cap (\hat{A}-\hat{B}) \text{ is infinite.} \tag{1.1}$$

Of course the major obstacle in meeting (1.1) is that we cannot verify in an r.e. way that an element of $U \cap A$ is actually an element of $U \cap (A-B)$. That is infinitely many elements x may appear to be in $U \cap (A-B)$ by virtue of there being a stage s such that $x \in U_s \cap (A_s - B_s)$ even though $U \cap (A-B) = \emptyset$. If we are not careful, this may cause us to enumerate infinitely many elements of $A-B$ into $\hat{U}$ thereby destroying (1.1). The technique of the Marker Lemma is essentially a method of assigning to each x a 'guess' at the cardinality of $U \cap (A-B)$. Suppose we are trying to 'guess' for instance whether $U \cap (A-B) \neq \emptyset$. Then we define a movable marker Λ whose position at stage s, $\Lambda(s)$, is given by

$$\Lambda(s+1) = \begin{cases} s & \text{if the least element of } U_s \cap (A_s - B_s) \neq \\ & \text{least element of } U_{s+1} \cap (A_{s+1} - B_{s+1}) \\ \Lambda(s) & \text{otherwise} \end{cases}$$

The important properties of Λ are that $\Lambda(s)$ is nondecreasing in s and $\lim_s \Lambda(s) < \infty$ iff $U \cap (A-B) \neq \emptyset$. Then the Marker Lemma is

Lemma 1.3 (Marker Lemma). *Let Λ be a movable marker such that the position of Λ at stage s, $\Lambda(s)$, is nondecreasing in s. Let $\{A_s\}_{s \in N}$ be an enumeration of an r.e. set A. If $x \in A$, let $t_x := \mu t(x \in A_t)$. Let $T := \{x : x < \Lambda(t_x)\}$. Suppose that $\lim_s \Lambda(s) = \infty$. Then*

(i) *$B \cup \bar{A}$ not r.e. $\Rightarrow T \cap (A-B)$ is infinite, and*

(ii) *$B \subset_m A \Rightarrow T \cap (A-B) =^* (A-B)$.*

Proof. Let $W = \{x : \exists s(x \notin A_s \text{ and } x < \Lambda(s))\}$. Then $W \supseteq \bar{A}$ since $\lim_s \Lambda(s) = \infty$. Now, in (i) since $B \cup \bar{A}$ is not r.e., $B \cup W \neq^* B \cup \bar{A}$ so that $W \cap (A-B)$ is infinite. But any $x \in W \cap (A-B)$ is also in $T \cap (A-B)$. For (ii), since $W \supseteq \bar{A}$, $W^* \supseteq \bar{B}$ so that $W \cap (A-B) =^* A-B$ and again $W \cap (A-B) \subseteq T \cap (A-B)$.

(All markers Λ used in our proofs will satisfy $\Lambda(s)$ is nondecreasing in s so the Marker Lemma will always be available.)

Now suppose x enters $\hat{A}$ at stage t_x. Then we assign to x a guess as to whether $U \cap (A-B) = \emptyset$ in the following way: if $x > \Lambda(t_x)$, x 'guesses' $U \cap (A-B) \neq \emptyset$ and if $x \leq \Lambda(t_x)$, x 'guesses' $U \cap (A-B) = \emptyset$. The Marker Lemma implies that if $\hat{B} \subset_m \hat{A}$, almost every $x \in \hat{A} - \hat{B}$ guesses correctly. (In fact, even if only $\hat{B} \cup \bar{\hat{A}}$ is not r.e., infinitely many elements of $\hat{A} - \hat{B}$ guess correctly.)

Now to meet the requirement (1.1) we need only have markers Λ_k which correspond to guesses that $|U \cap (A-B)| < k$. We define $\Lambda_k(s)$ so that $\lim_s \Lambda_k(s) < \infty$ iff $|U \cap (A-B)| \geq k$. We can arrange that $\Lambda_k(s)$ is increasing in k for each s and so we can let x guess that $|U \cap (A-B)| = k$ if $\Lambda_k(t_x) < x < \Lambda_{k+1}(t_x)$. The Marker Lemma implies that

$$\begin{aligned} &|U \cap (A-B)| = k \Rightarrow \text{almost every } x \in \hat{A} - \hat{B} \text{ guesses } |U \cap (A-B)| = k, \\ &|U \cap (A-B)| = \infty \Rightarrow \text{only finitely many elements of } A-B \text{ guess } |U \cap (A-B)| = k. \end{aligned} \tag{1.2}$$

(1.2) is enough to enumerate $\hat{U}$ to satisfy (1.1). For instance, enumerate $x \in \hat{U}$ if $x \in \hat{A}_s - \hat{B}_s$, and if x guesses $|U \cap (A-B)| = k$, then no other $y \in \hat{U}_s \cap (\hat{A}_s - \hat{B}_s)$ thinks $|U \cap (A-B)| = k$.

What the Marker Lemma really accomplishes is the following. Suppose $B \subset_m A$. For every Π_2 sentence P, we can assign to each x in A a guess as to the truth of P such that almost every x in $A-B$ guesses correctly. Further, this guess can be made early; i.e., at the stage x enters A.

2. Recursive sets and major subsets

Although our main theorem shows that major subsets B or A cannot be distinguished in $\mathscr{E}^*$ by $\mathscr{E}^*(A-B)$, there are a major number of lattice-definable properties which distinguish various major subsets of a given r.e. set A. In this

section, we give a construction of a large class of different major subsets of a fixed A. We are concerned here with the question of how recursive sets 'interact' with $\bar{B}$ whenever $B \subset_m A$. In this context, an alternate characterization of $\subset_m$ is useful.

Lemma 2.1. *$B \subset_m A$ if and only if for all recursive sets R, $R \subseteq^* A \Rightarrow R \subseteq^* B$.*

For any set X (not necessarily r.e.) let $\mathscr{E}_R(X) = \{R \cap X: R \text{ is recursive}\}$. Then $\mathscr{E}_R(X)$ is a Boolean algebra; let $\mathscr{E}_R^*(X)$ be $\mathscr{E}_R(X)$ modulo finite sets. If $B \subset_m A$, we wish to compare $\mathscr{E}_R^*(\bar{A})$ with $\mathscr{E}_R^*(A-B)$.

Fix for the remainder of this section a uniform enumeration of the recursive sets, $\{R_i\}_{i \in N}$. For instance

$$R_i = \{x: (\exists s)[x \in W_{i,s} \wedge \forall y > x (y \notin W_{i,s})]\}.$$

Then, under this indexing of the recursive sets, if X is d.r.e., $\mathscr{E}_R^*(X)$ is a Σ_3-Boolean algebra. That is, the relation $\{\langle i, j\rangle: R_i \cap X \subset^* R_j \cap X\}$ is Σ_3. The following lemma follows directly from Lemma 2.1.

Lemma 2.2. *Suppose that $B \subset_m A$. Then the map $R_i \cap \bar{A} \to R_i \cap (A-B)$ induces a homomorphism from $\mathscr{E}_R^*(\bar{A})$ to $\mathscr{E}_R^*(A-B)$. Further, since $\{i: R_i \cap (A-B) =^* \emptyset\}$ is Σ_3, the kernel of this homomorphism is a Σ_3-ideal $\mathscr{I}$ of $\mathscr{E}_R^*(\bar{A})$. ($\mathscr{I}$ is a Σ_3-ideal of $\mathscr{E}_R^*(X)$ if $\{i: R_i \cap X \in \mathscr{I}\}$ is Σ_3 and $\mathscr{I}$ is an ideal.)*

Our result is essentially that every Σ_3-ideal of $\mathscr{E}_R^*(\bar{A})$ gives rise to such a homomorphism. This is not quite true; if $R \subseteq \bar{A}$, then $R \cap (A-B) = \emptyset$ so that R must be in the ideal in question. But this is the only restriction.

Theorem 2.3. *Let A be an r.e. set. Let $\mathscr{I}$ be an ideal of $\mathscr{E}_R^*(\bar{A})$ such that $S = \{i: R_i \cap \bar{A} \in \mathscr{I}\}$ is a Σ_3 set and such that if $R \subseteq^* \bar{A}$, then $R \cap \bar{A} \in \mathscr{I}\}$. Then there is an r.e. set $B \subseteq A$ such that $R \cap (A-B) =^* \emptyset$ if and only if $R \cap \bar{A} \in \mathscr{I}$.*

Proof. We will suppose for convenience that S is actually a Π_2 set — it is easy to modify the following proof for Σ_3 sets by using an appropriate representation for Σ_3 sets. Therefore we may suppose that there is a recursive function f so that

$$i \in S \Leftrightarrow W_{f(i)} \text{ is infinite.} \tag{2.1}$$

For each $i \notin S$ we need to meet the requirements

$$N_{\langle i,j\rangle}: \quad |R_i \cap (A-B)| \geq j, \quad j \in N.$$

Since we cannot recognize that $i \notin S$ in a recursive way, we attempt to meet for each $i, j \in N$ the requirement $N_{\langle i,j\rangle}$. For this purpose we have markers $\Lambda_{\langle i,j\rangle}$ whose position at stage $s+1$ is defined by

$$\Lambda_{\langle i,j\rangle}(s+1) = \max \begin{cases} \Lambda_{\langle i,j\rangle}(s), & \\ x & \text{if } x \text{ is the } j\text{th element of } R_{i,s} \cap (A_s - B_s), \\ s & \text{if } |R_{i,s} \cap (A_s - B_s)| < j. \end{cases}$$

We suppose that we are given a simultaneous enumeration of all the r.e. sets mentioned above such that at most one element is enumerated in A at any stage s.

Construction

Stage $s+1$. *Step* 1. Suppose x is enumerated in A at stage $s+1$. Find the least pair $\langle i, j\rangle$ such that

$$x < \Lambda_{\langle i,j\rangle}(s), \quad \text{and} \quad x \in R_{i,s}.$$

If such a pair exists 'assign' x to $N_{\langle i,j\rangle}$. If not enumerate x in B.

Step 2. For each $y \in A_s - B_s$ such that $(\exists k)[W_{f(k),s+1} - W_{f(k),s}) \neq \emptyset$, y is assigned to $N_{\langle i,j\rangle}$, $k < \langle i, j\rangle$, and $y \in R_{k,s}]$ enumerate y in B.

Lemma 2.4. $\operatorname{Lim}_s \Lambda_{\langle i,j\rangle}(s) < \infty$ *iff* $N_{\langle i,j\rangle}$ *is satisfied.*

Lemma 2.5. *For each pair* $\langle i, j\rangle$, *only finitely many elements of* $A - B$ *are assigned to* $N_{\langle i,j\rangle}$.

Lemma 2.6. *If* $i \in S$, *then* $R_i \cap (A - B) =^* \emptyset$.

Lemma 2.7. *If* $i \notin S$, *then each requirement* $N_{\langle i,j\rangle}$ *is satisfied.*

Proof. Suppose to the contrary that for some fixed j, $\lim_s \Lambda_{\langle i,j\rangle}(s) = \infty$. Define an r.e. set W by

$$W = \{x : x \in R_i, (\forall k < \langle i, j\rangle)[k \in S \to x \notin R_k], \quad \text{and}$$
$$(\exists s)[x \in R_{i,s} \cap \bar{A}_s \quad \text{and} \quad x < \Lambda_{\langle i,j\rangle}(s)]\}.$$

Notice that W is actually recursive. Futher note that if $R = \bigcup \{R_k : k < \langle i, j\rangle, k \in S\}$, $R \cap \bar{A} \in \mathscr{I}$. (This of course uses the fact that $\mathscr{I}$ is an ideal.) Now $W \cap \bar{A} = (R_i - R) \cap \bar{A}$ and since $R_i \cap \bar{A}$ is not in $\mathscr{I}$, $W \cap \bar{A}$ is not in $\mathscr{I}$. Thus $W \not\subseteq^* \bar{A}$, and so $W \cap A$ is infinite. But it is easy to see that any $x \in W \cap A$ is assigned to a requirement N_l for some $l \leqslant \langle i, j\rangle$ and so never enters B. Thus $|W \cap (A - B)| = \infty$ so that $|R_i \cap (A - B)| = \infty$. This contradicts the hypothesis that $N_{\langle i,j\rangle}$ is not satisfied.

Lemmas 2.6 and 2.7 together guarantee that $R \cap (A - B) =^* \emptyset$ iff $R \cap \bar{A} \in \mathscr{I}$.

If A is recursive, then any ideal $\mathscr{I}$ satisfying the hypothesis of Theorem 2.3 must be improper and so the resulting B is equal to A. If A is nonrecursive and $\mathscr{I}$ is proper however, $B \subset_{\mathrm{m}} A$. The existence of such proper ideals $\mathscr{I}$ and hence the existence of major subsets is guaranteed by the next corollary.

Corollary 2.8 (Lachlan [2, Theorem 7]). *If* A *is nonrecursive, there is a set* B *such that* $B \subset_{\mathrm{m}} A$.

Proof. The sets $S_0 := \{i: R_i \cap \bar{A} =^* \emptyset\}$ and $S_1 := \{i: R_i \subseteq^* \bar{A}\}$ *are* Σ_3. Then

$$S = \{i: (\exists j, k)[j \in S_0, k \in S_1, \text{ and } R_i \subseteq R_j \cup R_k]\}$$

is Σ_3 and so $\mathscr{I} = \{R_i \cap \bar{A} \mid i \in S\}$ is a Σ_3-ideal of $\mathscr{E}^*(A)$. Thus there is a set $B \subseteq A$ so that

$$R_i \cap (A - B) =^* \emptyset \quad \text{iff} \quad i \in S.$$

Since A is nonrecursive, $N \notin \mathscr{I}$ so that $N \cap (A - B) = A - B$ is infinite. Thus $B \subset_{\mathrm{m}} A$ since $R_i \subseteq^* A \Rightarrow R_i \subseteq^* B$.

Lachlan's original proof of Corollary 2.8 was very different, his was an e-state construction. Lachlan constructed $A - B$ so that the eth-member of $A - B$ sought to maximize its e-state with respect to the array $\{U_e\}_{e \in N}$ where

$$U_e = \begin{cases} W_e & \text{if } W_e \supseteq \bar{A}, \\ \text{finite} & \text{if } W_e \not\supseteq \bar{A}. \end{cases}$$

Our original proof to Theorem 2.3 was based on this e-state construction of Lachlan's.

Lerman, Shore, and Soare introduce r-maximal major subsets in [4] to distinguish among sets such that $\mathscr{E}^*_{\mathrm{R}}(A) \cong \mathscr{B}$ where $\mathscr{B}$ is the countable atomless Boolean algebra.

Definition. $B \subset_{\mathrm{r}} A$ if $B \subseteq A$, $A - B$ is infinite and $A - B$ is r-cohesive, i.e., for every recursive R, $R \cap (A - B) =^* A - B$ or $R \cap (A - B) =^* \emptyset$.

They characterized the sets having r-maximal major subsets as those which have Δ_3-preference functions. This is easily seen to be equivalent to

Corollary 2.9 (Lerman, Shore, Soare [4, Theorem 1.2]). *A has an r-maximal major subset B iff there is a* Σ_3*-ideal* $\mathscr{I}$ *of* $\mathscr{E}^*_{\mathrm{R}}(\bar{A})$ *which is maximal and which contains each recursive R such that* $R \subseteq^* \bar{A}$.

If $B \subset_{\mathrm{rm}} A$, then B is 'close' to A. This is in accordance with Corollary 2.9 which characterizes B as arising from a maximal ideal of $\mathscr{E}^*_{\mathrm{R}}(\bar{A})$. Far removed from these are the major subsets $B \subset_{\mathrm{m}} A$, such that B arises from the smallest ideal consistent with the hypothesis of Theorem 2.3. In his decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$, Lachlan introduced major subsets which are 'small'.

Definition. B is a small subset of A, $B \subset_{\mathrm{s}} A$, if for every pair U, V of r.e. sets,

$$U \supseteq V \cap (A - B) \Rightarrow U \cup (V - A) \quad \text{is r.e.}$$

Corollary 2.10. *Let* $\mathscr{I}$ *be the smallest* Σ_3*-ideal of* $\mathscr{E}^*(\bar{A})$ *containing* $\{R: R \subseteq^* \bar{A}\}$.

(See the proof of Corollary 2.8 for the construction of $\mathscr{I}$). Then if $B \subset_{\mathrm{sm}} A$ we have $R \cap \bar{A} \in \mathscr{I}$ iff $R \cap (A-B) =^* \emptyset$.

Proof. Suppose that $R \cap (A-B) =^* \emptyset$. Let $U := R \cap (A-B)$ and $V := R$ in the definition of $B \subset_{\mathrm{s}} A$. Then we have that $V - A = R - A$ is r.e. Thus there is a recursive set S such that $S \cap \bar{A} = R \cap \bar{A}$ and $S \subseteq \bar{A}$. But then $R \cap \bar{A} \in \mathscr{I}$ which is what we desired to prove.

The condition of Corollary 2.10 is not an iff however. There are $B \subset_{\mathrm{m}} A$ such that B and A satisfy the condition of Corollary 2.10 but which do not satisfy $B \subset_{\mathrm{s}} A$.

3. The finite splitting property

In this section we show that if $B \subset_{\mathrm{m}} A$, $A - B$ has the finite splitting property. (See Definition 3.1 below.) Not only is this property essential to our proof, we think it is of interest in its own right. Let $\{W_e\}_{e \in N}$ be a standard enumeration of the r.e. sets.

Definition 3.1. Let $X \subseteq N$. Then X has the *finite splitting property* if there are total recursive functions f_0 and f_1 so that for every $e \in N$

(i) $W_{f_0(e)} \subseteq W_e$, $W_{f_1(e)} \subseteq W_e$,

(ii) $W_{f_0(e)} \cap W_{f_1(e)} = \emptyset$,

(iii) $W_{f_0(e)} \cup W_{f_1(e)} \supseteq X \cap W_e$,

(iv) $W_{f_1(e)} \cap X$ is finite,

(v) $W_e \cap X$ infinite $\Rightarrow$ $W_{f_1(e)} \cap X \neq \emptyset$.

If $X = \bar{B}$, then we say B has the *outer splitting property.*

The outer splitting property was introduced by Maass [5].

Theorem 3.2. *If $B \subset_{\mathrm{m}} A$, then $A - B$ has the finite splitting property.*

Proof. We suppose that we are given a simultaneous enumeration of the sets $\{W_e\}_{e \in N}$, A, and B. Given $e \in N$ we show how to enumerate $W_{f_0(e)}$ and $W_{f_1(e)}$. Let Λ_e be a movable marker whose position at stage s, $\Lambda_e(s)$, is defined by

$$\Lambda_e(0) = 0,$$

$$\Lambda_e(s+1) = \max \begin{cases} \Lambda_e(s), & \\ x & \text{if } x = \mu y (y \in (W_{f_1(e),s} - B_s)), \\ s & \text{if } W_{f_1(e),s} - B_s = \emptyset. \end{cases}$$

Obviously $\Lambda_e(s)$ is nondecreasing in s and $\lim_s \Lambda_e(s) < \infty$ if and only if $\exists x(x \in W_{f_1(e)} - B)$. At stage $s+1$, if $x \in (W_{e,s} \cap A_s) - (W_{f_0(e),s} \cup W_{f_1(e),s})$, we enumerate x in $W_{f_1(e)}$ if $x < \Lambda_e(s)$ and enumerate x in $W_{f_0(e)}$ if $x \geqslant \Lambda_e(s)$.

It is now clear that (i), (ii), (iii) and (iv) of Definition 3.1 are satisfied by f_0 and f_1. To see that (v) is satisfied, suppose for a contradiction that $W_e \cap (A-B)$ is infinite but $W_{f_1(e)} \cap (A-B) = \emptyset$. Then $\lim_s \Lambda_e(s) = \infty$ so that if $t_x = \mu s(x \in A_s)$, then $\{x: \Lambda(t_x) > x\} \,{}^*\!\supseteq A - B$ by the Marker Lemma. But any $x \in \{x: x < \Lambda(t_x)\}$ which is in W_e, will obviously be enumerated in $W_{f_1(e)}$ so that $W_{f_1(e)} \cap (A-B) =^* W_e \cap (A-B)$. This is a contradiction since it shows that $W_e \cap (A-B)$ is finite.

Corollary 3.3. *Suppose A is an r.e. set and g is a 1–1 recursive function mapping N onto A. Suppose $B \subset_m A$. Then $g^{-1}(B)$ has the outer splitting property.*

Corollary 3.3 gives an example of a set $\hat{B} = g^{-1}(B)$ which has the outer splitting property but such that $\mathscr{E}^*(\bar{\hat{B}}) \neq \mathscr{E}^*$. This gives an alternate proof to Lemma 2.4 of Maass [5]. There Maass shows that there is a set B with the outer splitting property such that $\bar{B}$ is not semilow$_{1.5}$. Since Maass has also showed that $\bar{B}$ is semilow$_{1.5}$ iff $\mathscr{E}(\bar{B})$ is effectively isomorphic to $\mathscr{E}^*$, the set $\hat{B}$ constructed above cannot be semilow$_{1.5}$. In fact, Maass has shown that for no major subset B, can B be semilow$_{1.5}$.

4. The construction

We give now the construction of the isomorphism from $\mathscr{E}^*(A-B)$ to $\mathscr{E}^*(\hat{A}-\hat{B})$ for $B \subset_m A$ and $\hat{B} \subset_m \hat{A}$. The verification that this construction succeeds occupies Section 5. The construction is self-contained but the reader might find it helpful to refer to other versions of the automorphism construction for various attempts at motivating the strategies involved. These are Maass [5], Soare [6, 7], and Stob [8]. We have, for instance, kept the standard numbering of rules and the usual names for auxiliary lists.

Fix then r.e. sets $B \subset_m A$ and $\hat{B} \subset_m \hat{A}$. Fix also a recursive function g which simultaneously enumerates the sets $\{W_e\}_{e \in N}$, $\{\hat{W}_e\}_{e \in N}$ where each of $\{W_e\}_{e \in N}$ and $\{\hat{W}_e\}_{e \in N}$ is a simultaneously r.e. sequence consisting of all the r.e. sets. We will assume that g enumerates every element of each of these sets infinitely often, that $W_0 = A$, $W_1 = B$, $\hat{W}_0 = \hat{A}$, and $\hat{W}_1 = \hat{B}$. If $x \in A$ $(x \in \hat{A})$, let t_x $(\hat{t}_x)$ be the first stage such that g enumerates x in $A = W_0$ $(\hat{A} = \hat{W}_0)$.

By Theorem 3.2, $A-B$ and $\hat{A}-\hat{B}$ have the finite splitting property. Fix recursive functions f_0, f_1 $(\hat{f}_0, \hat{f}_1)$ satisfying Definition 3.1 with respect to $\{W_e\}_{e \in N}$. We call $W_{f_1(e)}$ the critical part of W_e. During the construction we will be enumerating certain r.e. subsets of A $(\hat{A})$. By the recursion theorem, we may also assume we have indices (in the sequence $\{W_e\}_{e \in N}$ $(\{\hat{W}_e\}_{e \in N})$) for the critical parts of each of these sets.

For the construction we have as in Soare [6] two pinball machines M and $\hat{M}$. We fix two copies of the natural numbers, N and $\hat{N}$. The elements of N ($\hat{N}$) are used in machine M ($\hat{M}$) and we write $x, y, \ldots$ ($\hat{x}, \hat{y}, \ldots$) as variables for these elements.

On the side of machine M ($\hat{M}$) we construct r.e. sets $\{U_e\}_{e\in N}$, $\{\hat{V}_e\}_{e\in N}$ ($\{\hat{U}_e\}_{e\in N}$, $\{V_e\}_{e\in N}$). We write $U_{e,s}$ for the set of elements which are enumerated in U_e by the end of stage s of the construction, analogously for the other sets.

For any $x\in N$, any stage s, and any number e with $0\leqslant e\leqslant x$, we define

$$\nu(s, e, x) := \langle e, \{i\leqslant e \mid x\in U_{i,s}\}, \{i\leqslant e \mid x\in \hat{V}_{i,s}\}\rangle.$$

Similarly for $\hat{x}\in\hat{N}$ we set

$$\nu(s, e, \hat{x}) := \langle e, \{i\leqslant e \mid x\in \hat{U}_{i,s}\}, \{i\leqslant e \mid x\in V_{i,s}\}\rangle.$$

We use the symbol ν as a variable for triples $\langle e, \sigma, \tau\rangle$ where $e\geqslant 0$ is a natural number and σ, τ are subsets of $\{0, \ldots, e\}$. We call these triples states and we call $|\nu| := e$ the length of state $\nu = \langle e, \sigma, \tau\rangle$. For states $\nu = \langle e, \sigma, \tau\rangle$ and $\nu' = \langle e', \sigma', \tau'\rangle$ we define

$$\nu\preccurlyeq\nu' \ (\nu \text{ is an initial segment of } \nu') :\Leftrightarrow$$
$$e\leqslant e',\ \sigma = \sigma'\cap\{0, \ldots, e\} \text{ and } \tau = \tau'\cap\{0, \ldots, e\}.$$

We say that x ($\hat{x}$) has state ν at the end of stage s if $\nu\preccurlyeq\nu(s, x, x)$ ($\nu\preccurlyeq\nu(s, \hat{x}, \hat{x})$). We say that x ($\hat{x}$) has final state ν if $\nu = \lim_s \nu(s, x, x)$ ($\nu = \lim_s \nu(s, \hat{x}, \hat{x})$).

For states $\nu = \langle e, \sigma, \tau\rangle$, $\nu' = \langle e, \sigma', \tau'\rangle$ we define

$$\nu\geqslant\nu' \ (\nu \text{ covers } \nu') :\Leftrightarrow \sigma\supseteq\sigma' \text{ and } \tau\subseteq\tau',$$
$$\nu\geqslant_\sigma\nu' \ (\nu\ \sigma\text{-exactly covers } \nu') :\Leftrightarrow \sigma = \sigma' \text{ and } \tau\subseteq\tau',$$
$$\nu\geqslant_\tau\nu' \ (\nu\ \tau\text{-exactly covers } \nu') :\Leftrightarrow \sigma\supseteq\sigma' \text{ and } \tau = \tau'.$$

Observe that if $\nu\geqslant\nu'$ and if at some stage x is in state ν in machine M and $\hat{x}$ is in state ν' in machine $\hat{M}$, we can lift x and $\hat{x}$ into the same state of length $|\nu| = |\nu'|$ by enumerating x into some sets $\hat{V}_i$ and by enumerating $\hat{x}$ into some sets $\hat{U}_i$. If $\nu\geqslant_\sigma\nu'$ this can be done by just enumerating x into some $\hat{V}_i$ and if $\nu\geqslant_\tau\nu'$ this can be done by just enumerating $\hat{x}$ into some sets $\hat{U}_i$. These situations will become essential because it is our goal to have for every $e\in N$, $U_e =^* W_e\cap(A-B)$ and $V_e =^* \hat{W}_e\cap(\hat{A}-\hat{B})$ (so that we have nearly no freedom to enumerate elements into sets U_e, V_e or to leave them out of these sets as we like it) and to ensure simultaneously that for every state ν:

Infinitely many $x\in A-B$ have final state ν if and only if infinitely many $\hat{x}\in\hat{A}-\hat{B}$ have final state ν. (4.1)

We will show in Lemma 5.6 and Lemma 5.7 that this goal is achieved (Lemmas 5.1–5.5 are only needed in order to prove Lemma 5.6.) It is clear that (4.1) is enough to guarantee the existence of the desired isomorphism Φ from $\mathscr{E}^*(A-B)$

to $\mathscr{E}^*(\hat{A}-\hat{B})$. For we define

$$\Phi((U_e \cap (A-B))^*) = (\hat{U}_e \cap (\hat{A}-\hat{B}))^*$$

and

$$\Phi((V_e \cap (\hat{A}-\hat{B}))^*) = (\hat{V}_e \cap (A-B))^*.$$

Evidently Φ maps $\mathscr{E}^*(A-B)$ onto $\mathscr{E}^*(\hat{A}-\hat{B})$; (4.1) guarantees that Φ preserves $\subseteq^*$.

Pinball machine M (see Diagram 1) consists of hole H, tracks C and D, pockets P and Q and boxes B_ν in pocket P for every state ν. We write $B_{\nu,s}$ etc. for the set of elements which are at the respective place at the end of stage s. Pinball machine $\hat{M}$ is the same except that every notation and rule for $\hat{M}$ is written with a hat '^'.

The rules are indexed in the same way as the analogous rules in Soare [6].

Construction (we use the rules R_2, R_3, R_4, $\hat{R}_2$, $\hat{R}_3$, $\hat{R}_4$ in the construction which will be described subsequently)

Stage $s=0$: Do nothing.

Stage $s+1$: Adopt the first case which holds.

Case 1: Some element is on track C or D ($\hat{C}$ or $\hat{D}$). Apply R_3 ($\hat{R}_3$) if it is on track C ($\hat{C}$). Apply R_2 ($\hat{R}_2$) if it is on track D ($\hat{D}$).

Case 2: Some element is above hole H or $\hat{H}$. Take the least such element (if this is not unique take the one above H) and put it on track C ($\hat{C}$) if it was above hole H ($\hat{H}$).

Case 3: Otherwise.

We consider then one more value of the fixed enumeration function g.

(a) If g enumerates a new number into W_0 ($\hat{W}_0$), we enumerate this number into U_0 (V_0) and place it above hole H ($\hat{H}$) (we say that this number now enters machine M ($\hat{M}$)).

(b) If g enumerates a new number into W_1 ($\hat{W}_1$), then we remove this number from machine M ($\hat{M}$) and enumerate it into U_1 (V_1).

(c) If g enumerates a number x into W_e where $e>0$ such that x is not yet in W_0 (a number x into $\hat{W}_e$ where $e>0$ and x is not yet in $\hat{W}_0$), then go to stage $s+2$.

(d) If g enumerates a number $x>e$ into W_e where $e>1$ (a number $\hat{x}>e$ into $\hat{W}_e$ where $e>1$) which is not in U_e (V_e) and which sits at the moment in pocket Q ($\hat{Q}$) or in some box B_ν ($\hat{B}_\nu$) with $|\nu| \geq e$, then we remove this number from its present position, place it in hole H ($\hat{H}$) and enumerate it into U_e (V_e).

After we have followed the instructions in the adopted case we apply rule R_4 to all numbers in pocket Q in increasing order and we apply rule $\hat{R}_4$ to all numbers in pocket $\hat{Q}$ in increasing order. *End of the construction.*

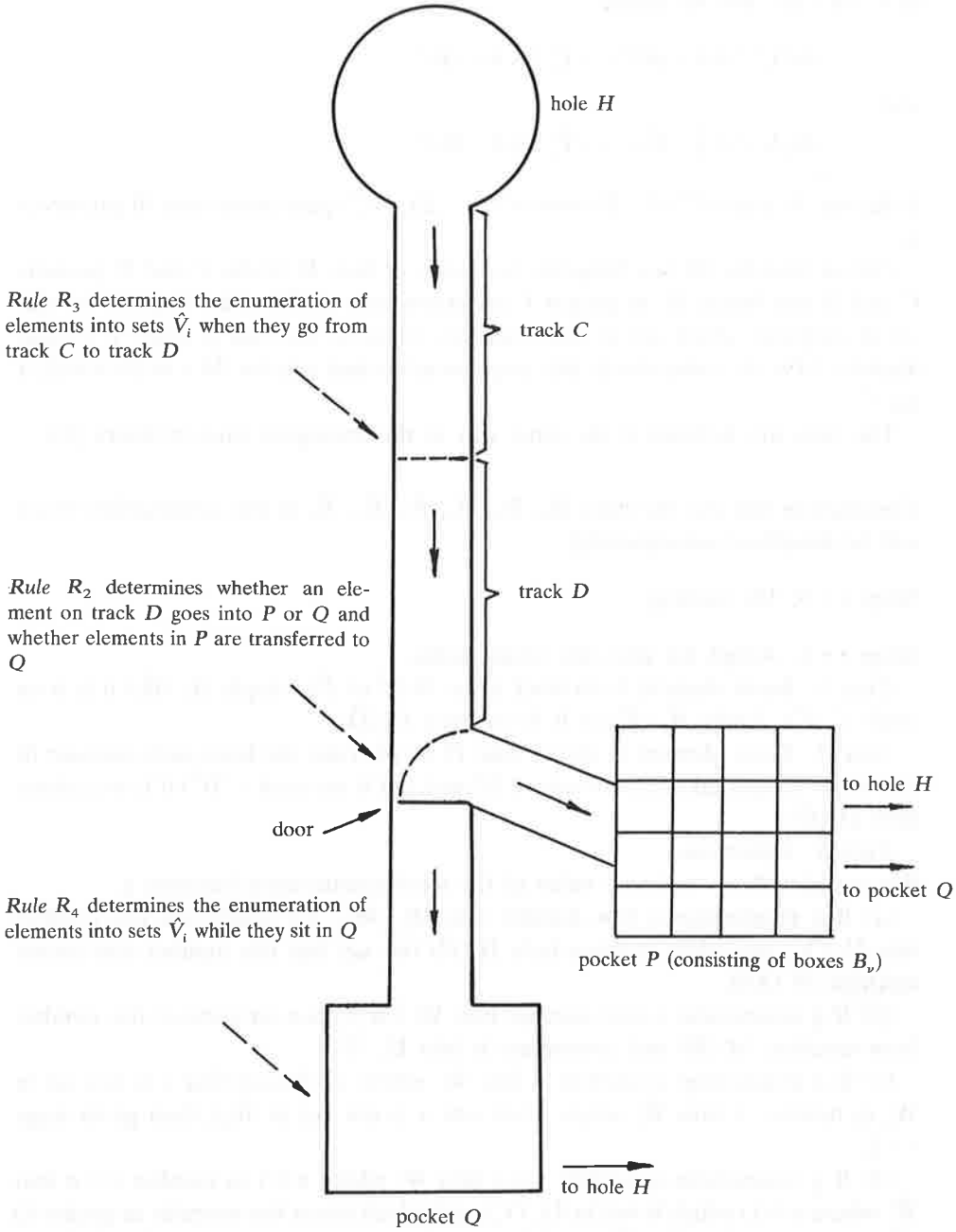

Diagram 1. Machine M

We now give the rules, defining also a number of markers, lists, and auxiliary sets. We will concentrate on the rules R_2, R_3 and $\hat{R}_4$, the rules $\hat{R}_2$, $\hat{R}_3$ and R_4 are completely analogous. R_2, R_3 and $\hat{R}_4$ are designed to prove that there are infinitely many x in pocket P in state ν at the end of the construction iff there are infinitely many $\hat{x}$ in pocket $\hat{Q}$ in state ν at the end of the construction. Rules $\hat{R}_2$, $\hat{R}_3$, and R_4 guarantee the same thing for pockets Q and $\hat{P}$.

If x is on track C at the end of stage s we define $\mathcal{S}_s(C)$ as the sequence of all states $\nu \leqslant \nu(s, x, x)$ (we say then that x *causes* $\nu \in \mathcal{S}_s(C)$). $\mathcal{S}_s(C)$ is empty if there is no element on track C at the end of stage s. $\mathcal{S}(C)$ is defined as the concatenation of all sequences $\mathcal{S}_s(C)$, $s \in N$.

Sequences $\mathcal{S}_s(D)$, $\mathcal{S}(D)$ are defined analogously. The sequence $\mathcal{S}_s(Q)$ consists of all states ν such that $\nu \leqslant \nu(s, x, x)$ for some x which is in pocket Q at the end of stage s but which was not yet in Q at the end of stage $s-1$.

We use X as a variable for tracks C, D and pocket Q.

For machine $\hat{M}$ we define $\mathcal{S}(\hat{C})$, $\mathcal{S}(\hat{D})$, $\mathcal{S}(\hat{Q})$ analogously and we use $\hat{X}$ as a variable for $\hat{C}$, $\hat{D}$, $\hat{Q}$.

We say that x ($\hat{x}$) causes $\nu \in \mathcal{S}(X)$ ($\mathcal{S}(\hat{X})$) if there is some $s \in N$ such that x ($\hat{x}$) causes $\nu \in \mathcal{S}_s(X)$ ($\mathcal{S}_s(\hat{X})$).

Define (in increasing order of $\leqslant$) a function q as follows. $q(s, \nu)$ is the least $y \in Q_s$ such that $\nu \leqslant \nu(s, y, y)$ and $q(s, \nu') \neq y$ for every $\nu' \prec \nu$. $q(s, \nu)$ is undefined if such a y does not exist.

Observe that this definition implies that for every $y \in Q_s$, there is an unique state ν with $y = q(s, \nu)$. We have $\nu \leqslant \nu(s, y, y)$ for this state ν.

Define the function $\hat{q}(s, \nu)$ for pocket $\hat{Q}$ in the same way.

Before giving the statements of each of the rules, we will first describe the strategy for meeting (4.1) and relate these to the pockets of the machines and the rules. The two strategies for insuring that infinitely many elements x of $A-B$ are in state ν iff infinitely many elements x of $A-B$ are in state ν are the following:

Strategy 1. Make sure that there is some $x > |\nu|$ in machine M in state ν that remains in $A-B$.

Strategy 2. Enumerate all elements of $\hat{A}-\hat{B}$ which are in state ν into states $\nu' > \nu$.

There are of course duals to each of these strategies. Strategy 1 is accomplished by Rule R_2 and results in placing elements in Box B_ν of pocket P. The key device which guarantees that this strategy succeeds when necessary is the finite splitting property. Thus Rule R_2 and the method of applying Strategy 1 are exactly the same in this construction as in Maass [5]. Rule R_2 will guarantee that if infinitely many elements of pocket $\hat{Q}$ are in state ν (and so that $\lim_s \hat{q}(s, \nu)$ exists) the box B_ν will get a stable element x, necessarily in state ν.

Define for states ν and $t \in N$

$$S_{\nu,t} := \{y: (\exists t' > t)[y \text{ causes } \nu \in \mathcal{S}_{t'}(D)]\}.$$

Rule R_2. Suppose x is on track D at the end of stage s. Let $s' < s$ be the last stage before s such that some element was on track D at stage s'. (If no such s' exists, let $s' := 0$.)

Step 1. For each ν such that $\hat{q}(\cdot, \nu)$ has not had a constant value since stage s', put each element of $B_{\nu,s}$ into pocket Q.

Step 2. For each ν such that $B_{\nu,s} = \emptyset$ or the smallest element of B_ν has changed since stage s', B_ν subscribes to all sets $S_{\nu',s}$ with $\nu \preccurlyeq \nu'$ and $|\nu'| \leqslant s$.

Step 3. Check whether there are ν and ν' such that $\nu \preccurlyeq \nu' \preccurlyeq \nu(s, x, x)$ and a stage $t < s$ such that B_ν had subscribed to the set $S_{\nu',t}$ and x is in the critical part of $S_{\nu',t}$. If such exist, choose ν of minimal length and put x in B_ν. If not, put x in pocket Q.

Rule $\hat{R}_2$ is analogous.

Strategy 2 places elements into pocket Q and is the responsibility of Rule R_4. Of course considerable care must be taken in executing strategy 2; that is in choosing states $\nu' > \nu$ to lift x's in state ν to. Strategy 2 played for the state ν merely shifts the burden of meeting (4.1) to the state ν'. Thus for each element $\hat{x}$ we will have a certain list $\mathcal{M}(\hat{x})$ of states ν' into which we allow $\hat{x}$ to be enumerated for the sake of strategy 2. The crucial property of the list $\mathcal{M}(\hat{x})$ will be the following which will be proved in Lemma 5.4. If infinitely many $x \in A - B$ cause $\nu \in \mathcal{S}(D)$ but only finitely many $x \in A - B$ cause $\nu' \in \mathcal{S}(D)$ for any $\nu' >_\tau \nu$, then $\nu \in \mathcal{M}(\hat{x})$ for almost every $\hat{x}$ of $\hat{A} - \hat{B}$. It is clear that such states ν represent desirable states from the point of view of strategy 2 since it is plausible that strategy 1 can be used successfully for such states ν. The list $\mathcal{M}(\hat{x})$ differs from its counterpart in other automorphism constructions in that it replaces the recursive approximations $\mathcal{M}_s$ to the $\mathcal{M}$ list which were used in previous constructions with uses of the marker lemma to associate with each $\hat{x}$ guesses as to which states should be on the list. The precise definition of the $\mathcal{M}$ list follows.

In order to define the lists $\mathcal{M}(\hat{x})$ we need the following markers $D^{\nu,k}(s)$, $3^\nu(s)$ and $2^{e,k}(s)$ for states ν and $e, k \in N$

$$D^{\nu,k}(0) := k,$$

$$D^{\nu,k}(s+1) = \begin{cases} \max\{k, s+1\} & \text{if } G_{\nu,s+1} \text{ has less than } k \text{ elements or if the first } k \text{ elements of } G_{\nu,s+1} \text{ are not the first } k \text{ elements of } G_{\nu,s}, \\ D^{\nu,k}(s) & \text{otherwise} \end{cases}$$

where $G_{\nu,t}:=\{y \mid y\notin B_t$ and y has caused $\nu\in\mathcal{S}_{t'}(D)$ for some $t'\leqslant t\}$.

$$3^\nu(0):=0,$$

$$3^\nu(s+1):=\begin{cases} s+1 & \text{if } (\exists\nu'\leqslant\nu)[(B_{\nu',s+1}=\emptyset \text{ or the smallest elements of } B_{\nu',s+1} \text{ is not the smallest element of } B_{\nu',s}) \text{ and } (\forall\nu''\leqslant\nu')[(\hat{q}(\cdot,\nu'') \text{ has had a constant value since stage } |\nu|]], \\ 3^\nu(s) & \text{otherwise.}\end{cases}$$

The marker $2^{e,k}(s)$ has to be defined simultaneously with the lists $\mathcal{M}(\hat{x})$ for $\hat{x}\in A_s$. We set for every $\hat{x}\in\hat{\mathcal{M}}$:

$$\mathcal{P}(\hat{x}):=\{\nu \mid \exists\nu'\in\mathcal{M}(\hat{x})(\nu'\geqslant_\tau\nu)\}.$$

We say that $\hat{x}$ causes $\nu\in\mathcal{S}_t(\hat{X})-\mathcal{P}(\mathcal{S}(\hat{X})-\mathcal{P})$ if $\hat{x}$ causes $\nu\in\mathcal{S}_t(\hat{X})$ $(\mathcal{S}(\hat{X}))$ and $\nu\notin\mathcal{P}(\hat{x})$.

$$2^{e,k}(0):=k,$$

$$2^{e,k}(s+1):=\begin{cases} \max\{k,s+1\} & \text{if } H_{e,s+1} \text{ has less than } k \text{ elements or if the first } k \text{ elements of } H_{e,s+1} \text{ are not the first } k \text{ element of } H_{e,s}, \\ 2^{e,k}(s) & \text{otherwise}\end{cases}$$

where

$$H_{e,t}:=\{\hat{y} \mid \hat{y}\notin\hat{B}_t \text{ and there is some state } \nu \text{ with } |\nu|\leqslant e \text{ and some stage } t'\leqslant t \text{ such that } \hat{y} \text{ causes } \nu\in\mathcal{S}_{t'}(\hat{X})-\mathcal{P} \text{ for some } \hat{X}\}.$$

For $\hat{x}\in\hat{A}_s$ we define

$$\mathcal{M}(\hat{x}):=\{\nu \mid |\nu|\leqslant\hat{x} \text{ and } 3^\nu(t_{\hat{x}})<\hat{x} \text{ and for } k_{\hat{x}}:=\max\{k \mid (\exists e<|\nu|)[2^{e,k}(t_{\hat{x}})<\hat{x}]\} \text{ we have } D^{\tilde{\nu},k_{\hat{x}}}(t_{\hat{x}})<\hat{x} \text{ for every } \tilde{\nu}\leqslant\nu\}.$$

Further for $\hat{x}\in\hat{A}_s$ we define

$$d(s,\hat{x}):=\max(\{-1\}\cup\{e\geqslant 0 \mid \nu(s,e,\hat{x})\in\mathcal{P}(\hat{x})\}).$$

We now explain what the markers in the above definition are trying to accomplish. Markers $D^{\nu,k}$ simply provide each $\hat{x}\in\hat{A}$ with a guess as to whether k elements of $A-B$ cause $\nu\in\mathcal{S}(D)$. Such ν's are added to list $\mathcal{M}(\hat{x})$ unless they are excluded for some reason. The exclusions are governed by the markers 3^ν and $2^{e,k}$ and are exactly analogous to the clause III and II exclusions of Soare [7] and Stob [8]. The marker 3^ν causes exclusion from $\mathcal{M}(\hat{x})$ of states ν for which strategy 1 is failing for ν; i.e., box B_ν is not getting a stable element. The markers $2^{e,k}$ cause exclusions of states ν of length $>e$ if it appears that there are states of length $\leqslant e$ for which strategy 2 is failing. This happens when there are elements $\hat{x}$ of $\hat{A}-\hat{B}$ in states $\nu'\notin\mathcal{P}(\hat{x})$. If $\hat{x}$ is in such a state ν', there is no state $\nu\in\mathcal{M}(\hat{x})$ of the same

length as ν' which $\hat{x}$ can be enumerated into. (This is the purpose of the definition of $\mathcal{P}(\hat{x})$.) Thus, the definition of $\mathcal{M}(\hat{x})$ amounts to the following. $\nu \in \mathcal{M}(\hat{x})$ if $\hat{x}$ guesses that ν is not excluded from $\mathcal{M}$ by clause 3 exclusion and x guesses that more elements of $A-B$ have caused $\nu \in \mathcal{S}(D)$ than elements of $\hat{A}-\hat{B}$ have caused ν to be excluded from $\mathcal{M}$ by clause 2 exclusion.

Rule $\hat{R}_4$ simply selects a state $\nu \in \mathcal{M}(\hat{x})$ into which to enumerate $\hat{x}$ if $\hat{x}$ is to be placed in pocket Q.

Rule $\hat{R}_4$. If $\hat{x}$ has entered pocket $\hat{Q}$ after the end of stage s and if $d(s, \hat{x}) \geq 0$, we take a state $\nu \in \mathcal{M}(\hat{x})$ with $\nu \geq_\tau \nu(s, d(s, \hat{x}), \hat{x})$ such that $k_\nu := \max\{k \mid D^{\nu,k}(t_{\hat{x}}) < \hat{x}\}$ is as large as possible. (If this doesn't determine ν uniquely we take the alphabetically first such ν.) We then enumerate $\hat{x}$ into sets $\hat{U}_i$ with $i \leq |\nu|$ (so that $\hat{x}$ gets into state ν).

Rule R_3 enumerates certain elements x in machine M into sets V_i. The major goal of Rule R_3 is to guarantee claim 1 of Lemma 5.5. Rule R_3 should be viewed as a Rule which repairs the injuries caused to requirement (4.1) for ν by virtue of elements $\hat{y}$ of $\hat{A}-\hat{B}$ causing states $\nu \in \mathcal{S}(\hat{X}) - \mathcal{P}$. This was also the purpose of clause 2 exclusion described above. A secondary purpose of Rule R_3 is to insure that if infinitely many elements of $A-B$ occur on track C in state ν, infinitely many elements of $A-B$ occur on track D in state ν. This is verified in Lemma 5.3 and used in Lemma 5.4. The markers $Y^{\nu,k}$ below provide elements of A with guesses as to which states ν have been injured k times. Elements x which guess that such injuries occur are enumerated in states $\nu' \geq_\tau \nu$ whenever possible. The markers $Z^{\nu',k}$ are used to restrain this enumeration by requiring of elements x so enumerated that they guess that many elements of $\hat{A}-\hat{B}$ are in fact in state ν'.

In order to give Rule R_3 we have to define markers $Y^{\nu,k}(s)$, $Z^{\nu,k}(s)$ for states ν and $k \in N$.

$$Y^{\nu,k}(0) := k,$$

$$Y^{\nu,k}(s+1) := \begin{cases} \max\{k, s+1\} & \text{if } E_{\nu,s+1} \text{ has less than } k \text{ elements} \\ & \text{or if the first } k \text{ elements of } E_{\nu,s+1} \text{ are} \\ & \text{not the first } k \text{ elements of } E_{\nu,s}, \\ Y^{\nu,k}(s) & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} E_{\nu,t} := \{\hat{y} \mid \hat{y} \notin \hat{B}_t \text{ and there are stages } t'' \leq t' \leq t \text{ such that} \\ \nu = \nu(t'', |\nu|, \hat{y}) \text{ and } \hat{y} \text{ causes for some } \hat{X} \\ \nu(t', |\nu|, \hat{y}) \in \mathcal{S}_{t'}(\hat{X}) - \mathcal{P}\}. \end{aligned}$$

$$Z^{\nu,k}(0) := k,$$

$$Z^{\nu,k}(s+1) := \begin{cases} \max\{k, s+1\} & \text{if } F_{\nu,s+1} \text{ has less than } k \text{ elements or the} \\ & \text{first } k \text{ elements of } F_{\nu,s+1} \text{ are not} \\ & \text{the first } k \text{ elements of } F_{\nu,s}, \\ Z^{\nu,k}(s) & \text{otherwise} \end{cases}$$

where

$$F_{\nu,t} := \{\hat{y} \mid \hat{y} \notin \hat{B}_t \text{ and } \hat{y} \text{ has caused } \tilde{\nu} \in \mathcal{S}_{t'}(\hat{C}) \text{ for some state } \tilde{\nu} \leqslant \nu \text{ and some stage } t' \leqslant t\}.$$

Rule R_3 depends on certain sets T_n. We first describe how to enumerate the sets T_n. Notice that for any set T which we enumerate, if we have enumerated x in T we can assume that we know whether x is in the critical part of T.

Fix a recursive function h which enumerates the set

$$\{\langle \nu, k, i\rangle : \nu \text{ is a state}, k \in N, i \in \{0, 1\}\}.$$

Suppose that x is on track C at stage s. By induction on n we say whether to enumerate x in T_n at stage s. There are two cases:

$h(n) = \langle \nu, k, 0\rangle$: Enumerate x in T_n if $n < s$, x is not yet in the critical part of $T_{\tilde{n}}$ for any $\tilde{n} < n$, and $\exists \nu' = \langle e, \sigma', \tau'\rangle \geqslant \nu = \langle e, \sigma, \tau\rangle$, such that

(i) x causes $\nu' \in \mathcal{S}_s(C)$,

(ii) $Y^{\nu,k}(t_x) < x$, and

(iii) $Z^{\tilde{\nu},n}(t_x)$ for every $\tilde{\nu} \leqslant \langle e, \sigma', \tau\rangle$.

$h(n) = \langle \nu, k, 1\rangle$: Enumerate x in T_n if $n < s$, x is not yet in the critical part of $T_{\tilde{n}}$ for any $\tilde{n} < n$, and x causes $\nu \in \mathcal{S}_s(C)$.

Notice that for any x, x is in the critical part of only finitely many sets T_n.

Rule R_3. Suppose x is on track C at the end of stage s. At stage s, according to the above enumeration procedure, x will be enumerated in at most one set T_n such that x is in the critical part of T_n. If there is no such n or if $h(n) = \langle \nu, k, 1\rangle$ for some ν and k, place x on track D. If $h(n) = \langle \nu, k, 0\rangle$, then enumerate x in sets $\hat{V}_i$ for $i \leqslant |\nu|$ so that the state of x becomes $\geqslant_\tau \nu$ and then place x on track D. (This enumeration is possible because of the conditions for x to be enumerated in T_n.)

5. The verification

A trivial proof by induction on the enumeration given by the function g shows the following. Every $x \in A$ enters hole H at some stage but no number remains forever in hole H. At every stage there is at most one element on one of the tracks C, D, $\hat{C}$, $\hat{D}$. This number is moved downwards at the next stage. Further $x \in A$ can move upwards in machine M (i.e. from P or Q into hole H) only if x is enumerated in some new U_i with $i \leqslant x$. No number x jumps directly from one box in P to another (although x may be recycled to H and get into a different box when it enters P the time). Therefore every number $x \in N$ moves only finitely often in machine M and is either permanently removed from M (if $x \in B$) or sits from some stage on permanently in Q or in a box B_ν in P (if $x \in A - B$). The same holds for elements $\hat{x} \in \hat{N}$ in machine $\hat{M}$. For permanent residents of Q we have the following.

Lemma 5.1. (i) *For every permanent resident x of Q there is a unique state ν such that $x = \lim_s q(s, \nu)$. This state ν satisfies $\nu \leqslant \lim_s \nu(s, x, x)$.*

(ii) *For every ν, if $\lim_s q(s, \nu)$ exists, then so does $\lim_s q(s, \nu')$ for each $\nu' \prec \nu$.*

Proof. (i) Assume that there is a permanent resident x of Q such that for no ν, $x = \lim_s q(s, \nu)$. Let x be minimal with this property. Let s_0 be a stage such that for every $y < x$ with $y \in Q_s$ for some $s \geqslant s_0$ there is a state ν_y with $\forall s \geqslant s_0$ $(y = q(s, \nu_y))$. Further we assume that $\nu(s, x, x)$ is constant after stage s_0. Let ν_0 be of minimal length so that $x = q(s, \nu_0)$ for some $s > s_0$. Since by assumption we have $x \neq \lim_s q(s, \nu)$ there is some $s_1 > s_0$ such that $x = q(s_1, \nu_0) \neq q(s_1 + 1, \nu_0) = y$ for some y. Then $y < x$ according to the definition of q and this contradicts the choice of s_0.

(ii) Assume that $x = q(s, \nu)$ for all $s \geqslant s_0$. By (i) there is some $s_1 > s_0$ such that for every $y < x$ with $(\exists s \geqslant s_1)[y \in Q_s]$ there is some state ν_y with $(\forall s \geqslant s_1)$ $y = q(s, \nu_y)]$. Consider some $\nu' \prec \nu$. For every $s \geqslant s_1$, $q(s, \nu')$ is defined and less than x by the definition of q. Therefore $\nu' = \nu_y$ for some $y < x$.

For Lemma 5.1 as well as each of the remaining lemmas there is a dual whose proof we omit. Everthing is symmetric.

Lemma 5.2. *For every s, ν, and $k < k'$, $D^{\nu,k}(s) \leqslant D^{\nu,k'}(s)$. Likewise for the markers $2^{e,k}$, $Y^{\nu,k}$, and $Z^{\nu,k}$.*

Proof. This is immediate from the definition. If $D^{\nu,k}$ changes its position at stage $s+1$, (necessarily so that $D^{\nu,k}(s+1) = s+1$), then so does $D^{\nu,k'}$.

Lemma 5.3. *Assume that infinitely many elements of $A - B$ cause $\nu \in \mathcal{S}(C)$. Then infinitely many elements of $A - B$ cause $\nu \in \mathcal{S}(D)$.*

Proof. Fix n such that $h(n) = \langle \nu, k, 1 \rangle$. We first show under the hypothesis that infinitely many elements of $A - B$ cause $\nu \in \mathcal{S}(C)$, that $T_n \cap (A - B)$ is infinite. For if $x \in A - B$ causes $\nu \in \mathcal{S}(C)$ say at stage s, then x is enumerated in T_n at stage s unless x is in the critical part of some $T_{\tilde{n}}$, $\tilde{n} < n$. Since only finitely many $x \in A - B$ can be in the critical part of any $T_{\tilde{n}}$, $\tilde{n} < n$, $T_n \cap (A - B)$ is infinite. Thus the critical part of T_n contains an element $x_n \in A - B$ by the finite splitting property. Such an element x_n must be in T_n because there was a stage s such that x_n was on track C in state ν at stage s. Then by Rule R_3, at stage $s+1$, x_n is placed on track D in state ν. Now the elements x_n for $h(n) = \langle \nu, k, 1 \rangle$, $k \in N$, form an infinite set for each x_n may be in the critical part of only finitely many $T_{\tilde{n}}$. Thus infinitely many elements of $A - B$ cause $\nu \in \mathcal{S}(D)$.

Lemma 5.4. *Fix ν. Assume there is a $k_0 \in N$ such that for every $e' < |\nu|$, $\lim_s 2^{e',k_0}(s) = \infty$. Assume also that infinitely many elements of $A - B$ cause $\nu \in \mathcal{S}(D)$ but only finitely many elements of $A - B$ cause $\nu' \in \mathcal{S}(D)$ for any ν' such that $\nu' \,{}_{\tau}{>}\, \nu$. Then $\nu \in \mathcal{M}(\hat{x})$ for almost every $\hat{x} \in \hat{A} - \hat{B}$.*

Proof. We will first show that $\lim_s 3^\nu(s) < \infty$. To do this, suppose that $\nu' \leqslant \nu$ which satisfies

$$(\forall \nu'' \leqslant \nu')[\hat{q}(\cdot, \nu'') \text{ has a constant value since stage } |\nu|]. \tag{5.1}$$

We will show that (5.1) implies that box $B_{\nu'}$ gets a stable element (i.e., an element x which is in box $B_{\nu'}$ for cofinitely many stages). For suppose that ν' is of minimal length such that the hypothesis of (5.1) holds but box $B_{\nu'}$ does not get a stable element. Then for infinitely many s, $B_{\nu'}$ subscribes to the set $S_{\nu,s}$ in step 2 of Rule R_2. Since infinitely many elements of $A-B$ cause $\nu \in \mathcal{S}(D)$, each set $S_{\nu,s}$ contains infinitely many elements of $A-B$ so that the critical part of each $S_{\nu,s}$ contains an element of $A-B$. Since every element belongs to only finitely many of these sets, we have that $S := (A-B) \cap \{x : x \text{ is in the critical part of some } S_{\nu,s} \text{ to which } B_{\nu'} \text{ has subscribed}\}$ is infinite.

Now by the minimality of ν', each box $B_{\nu''}$, $\nu'' \prec \nu'$ gets a stable element. Therefore, each of these subscribes to only finitely many sets and only finitely many elements of $A-B$ are placed in box $B_{\nu''}$. Thus, almost every element of S is placed in box B_ν' in state ν. By assumption, all these numbers leave box B_ν' at some later stage. Since $\lim_s \hat{q}(s, \nu')$ exists, almost all of these elements must leave box B_ν' because they are placed above hole H and enumerated in some new set U_e, $e \leqslant |\nu'|$ according to case 3(d) of the construction. Thus, each of these elements must later come on track C in some stage $\tilde{\nu}_\tau > \nu$. Fix some $\tilde{\nu}$ such that $\tilde{\nu}_\tau > \nu$ and infinitely many elements of S come on track C in state $\tilde{\nu}$. Then, according to Lemma 5.3, infinitely may elements of $A-B$ come on track D in state $\tilde{\nu}$. This contradicts the hypothesis of the lemma so no such ν' exists. Now it is easy to see from the definition that $\lim_s 3^\nu(s) < \infty$.

Now since infinitely many elements of $A-B$ cause $\nu \in \mathcal{S}(D)$, we have that $\lim_s D^{\tilde{\nu},k_0}(s) < \infty$ for each $\tilde{\nu} \leqslant \nu$. Thus almost every element $\hat{x}$ of $\hat{A}-\hat{B}$ satisfies

$$|\nu| \leqslant \hat{x} \quad \text{and} \quad 3^\nu(t_{\hat{x}}) < \hat{x} \quad \text{and} \quad (\forall \tilde{\nu} \leqslant \nu)[D^{\tilde{\nu},k_0}(t_{\hat{x}}) < \hat{x}]. \tag{5.2}$$

Further, almost every $\hat{x} \in \hat{A}-\hat{B}$ satisfies

$$(\forall e' < |\nu|)[2^{e'k_0}(t_{\hat{x}}) \geqslant \hat{x}] \tag{5.3}$$

because of the Marker Lemma. Using Lemma 5.2, (5.3) implies that

$$(\forall e' < |\nu|)[(\forall k \geqslant k_0)[2^{e',k}(t_{\hat{x}}) \geqslant \hat{x}]].$$

Therefore, for almost every $\hat{x} \in \hat{A}-\hat{B}$ we have

$$k_{\hat{x}} := \max\{k : (\exists e' < |\nu|)[2^{e,k}(t_{\hat{x}})\} < \hat{x}]\} < k_0$$

and so that

$$(\forall \tilde{\nu} \leqslant \nu)[D^{\tilde{\nu},k_x}(t_{\hat{x}}) < \hat{x}]. \tag{5.4}$$

Together, (5.2) and (5.4) directly imply that $\nu \in \mathcal{M}(\hat{x})$ so that $\nu \in \mathcal{M}(\hat{x})$ for almost every $\hat{x} \in \hat{A}-\hat{B}$.

Lemma 5.5. (i) *For every state ν, only finitely many elements of $\hat{A}-\hat{B}$ cause $\nu \in \mathcal{S}(\hat{X})-\mathcal{P}$ for some $\hat{X}$.*

(ii) *For every state ν only finitely many elements of $A-B$ cause $\nu \in \mathcal{S}(X)-\mathcal{P}$ for some X.*

Proof. We prove (i) and (ii) simultaneously by induction $|\nu|$. Assume then that (i) and (ii) hold for all ν with $|\nu|<e$. Note that this implies directly that there is a $k_0 \in N$ such that $\lim_s 2^{e',k_0}(s)=\infty$ for all $e'<e$.

Assume for a contradiction that (i) does not hold for some state ν_1 of length e. Fix infinitely many $\hat{y}_j \in \hat{A}-\hat{B}$ and stages t_j, $j \in N$, such that for every j there is an $\hat{X}$ such that

$$\hat{y}_j \quad \text{causes} \quad \nu_1 \in \mathcal{S}_{t_j}(\hat{X})-\mathcal{P}. \tag{5.5}$$

Let

$$\mathcal{T}_j := \{\nu : (\exists s \leqslant t_j)[\hat{y}_j \in \hat{M}_s \text{ and } \nu(s, e, \hat{y}_j)=\nu]\}.$$

Let $\mathcal{T}$ be the concatenation of the sets $\mathcal{T}_j$ (as finite sequences).

Claim 1. If ν occurs infinitely often in $\mathcal{T}$ and if infinitely many elements of $A-B$ cause $\nu' \in \mathcal{S}(C)$ for some $\nu' \geqslant \nu$, then infinitely many elements of $A-B$ cause $\nu'' \in \mathcal{S}(D)$ for some $\nu'' \geqslant_\tau \nu$.

Proof. Let $\nu := \langle e, \sigma, \tau\rangle$ and let $\nu' := \langle e, \sigma', \tau'\rangle \geqslant \nu$ be such that infinitely many elements of $A-B$ cause $\nu' \in \mathcal{S}(C)$.

Suppose that $\hat{y}_j$ causes an occurrence of ν in $\mathcal{T}$ at stage v_j. Then we claim that there is a stage s_j such that $\hat{y}_j$ was on track C at stage s_j was on track C at stage s_j in a state $\nu_0 \leqslant \nu$. For if $\hat{y}_j$ was above hole H at stage v_j then $\hat{y}_j$ later appears on track $\hat{C}$ in the very same state ν. On the other hand, if $\hat{y}_j$ was on some other track at stage v_j then y_j was on track $\hat{C}$ at some stage $s_j \leqslant v_j$ in stage $v_0 \leqslant_\tau \nu$. Since $\nu \leqslant \langle e, \sigma, \tau\rangle$, we have that

$$\lim_s Z^{\tilde{\nu},k}(s)<\infty \quad \text{for every } \tilde{\nu} \leqslant \langle e, \sigma', \tau\rangle,\ k \in N. \tag{5.6}$$

Now (5.5) guarantees that

$$\lim_s Y^{\nu,k}(s)<\infty \quad \text{for every } k \in N. \tag{5.7}$$

Now fix k and let $n_k = \langle \nu, k, 0\rangle$. For every $\tilde{n}<n_k$, the critical part of $T_{\tilde{n}}$ contains only finitely many elements of $A-B$. Thus by (5.6) and (5.7) and the definition of T_{n_k}, almost every $x \in A-B$ such that x causes $\nu' \in \mathcal{S}(C)$ for some $\nu' \geqslant \nu$ is an element of T_{n_k}. Thus the critical part of T_{n_k} contains an element x_k of $A-B$. As k ranges over N, there are infinitely many such x_k's, since every x can be in the critical part of only finitely many T_n's. Now according to Rule R_3, each x_k is placed on track D in some state $\nu''_\tau \geqslant \nu$ at the stage following the stage it was put in T_n. This proves Claim 1.

Claim 2. If ν occurs infinitely often in $\mathcal{T}$ and there is a $\nu' \geqslant \nu$ such that infinitely many elements of $A-B$ cause $\nu' \in \mathcal{S}(C)$ then $\nu \in \mathcal{P}(\hat{x})$ for almost all $\hat{x} \in \hat{A}-\hat{B}$.

Proof. By Claim 1 there is a $\nu''_{\tau} \geqslant \nu$ such that infinitely many elements of $A-B$ cause $\nu'' \in \mathcal{S}(D)$. Choose ν' with this property but so that no $\nu''_{\tau} \geqslant \nu'$ has the same property. Then $\nu' \in \mathcal{M}(\hat{x})$ and so $\nu \in \mathcal{P}(\hat{x})$ for almost every $\hat{x}$.

Claim 3. If ν occurs infinitely often in $\mathcal{T}$, then infinitely may elements of $A-B$ cause $\nu' \in \mathcal{S}(C)$ for some $\nu' \geqslant \nu$.

Proof. By contradiction fix $\nu_2 := \langle e, \sigma_2, \tau_2 \rangle$ such that the claim fails for ν_2 and σ_2 is minimal and τ_2 is minimal for σ_2. Of course $\sigma_2 \neq \emptyset$ since $A-B$ is infinite and each element of $A-B$ passes over track C at some stage.

Fix an infinite set $J \subseteq N$, a state $\nu_3 = \langle e, \sigma_3, \tau_3 \rangle$ and stages $s_j \leqslant v_j$ $(j \in J)$ so that for every $j \in J$.

$$\nu(s_j - 1, e, \hat{y}_j) = \langle e, \sigma_3, \tau_3 \rangle \neq \langle e, \sigma_2, \tau_2 \rangle = \nu(s_j, e, \hat{y}_j).$$

By the minimality of ν_2, the claim holds for ν_3. Thus $\sigma_3 \subsetneq \sigma_2$ since otherwise the claim would hold for ν_2. Thus there is an infinite set $J' \subseteq J$ such that either

for every $j \in J'$, Rule $\hat{R}_3$ is applied to $\hat{y}_j$ at stage s_j, or

for every $j \in J'$, Rule $\hat{R}_4$ is applied to $\hat{y}_j$ at stage s_j.

Case 1. For every $j \in J'$, Rule $\hat{R}_3$ is applied to $\hat{y}_j$ at s_j.

By (ii) of the induction hypothesis and the definition of the markers $Y^{\tilde{\nu},k}$, there is a k_1 such that if $|\tilde{\nu}| < e$, $\lim_s \hat{Y}^{\tilde{\nu},k_1}(s) = \infty$. Then by Lemma 5.2 and the Marker Lemma, for almost every $\hat{x} \in \hat{A}-\hat{B}$ we have

$$(\forall\, |\tilde{\nu}| < e)(\forall k \geqslant k_1)[\hat{Y}^{\tilde{\nu},k}(t_{\hat{x}}) \geqslant \hat{x}]. \tag{5.8}$$

Then (5.8) implies that only finitely may elements of $\hat{A}-\hat{B}$ can be in the critical part of any set T_n such that $h(n) = \langle \tilde{\nu}, k, 0 \rangle$, $|\nu'| < e$ and $k \in N$. This in turn implies that for almost every $j \in J'$ there is a $n_j \in N$ with $h(n_j) = \langle \nu_j, k_j, 0 \rangle$, $|\nu_j| \geqslant e$ and $k \in N$ and y_j is in the critical part of T_{n_j}. By the definition of T_{n_j} we have that for each of these j

$$\hat{Z}^{\nu_2, n_j}(t_{\hat{y}_j}) < \hat{y}_j.$$

The critical part of each T_{n_j} is finite so that for each n_0 there are infinitely many $j \in J'$ such that

$$\hat{Z}^{\nu_2, n_0}(\hat{t}_{y_j}) \leqslant \hat{Z}^{\nu_2, n_j}(\hat{t}_{y_j}) < \hat{y}_j.$$

By the Marker Lemma, we then must have

$$\lim_s \hat{Z}^{\nu_2, n}(s) < \infty \quad \text{for every } n \in N. \tag{5.9}$$

But (5.9) and the definition of the markers $\hat{Z}^{\nu_2, n}$ imply that infinitely many elements of $A-B$ cause $\nu' \in \mathcal{S}(C)$ for some $\nu' \geqslant \nu_2$. This contradicts our choice of ν_2.

Case 2. For every $j \in J'$, Rule $\hat{R}_4$ is applied to $\hat{y}_j$ at s_j.

Since the claim holds for ν_3 and ν_3 occurs in $\mathcal{T}$ infinitely often, Claim 2 implies that $\nu_3 \in \mathcal{P}(\hat{x})$ for almost every $\hat{x} \in \hat{A} - \hat{B}$. This implies that

$$d(s_j - 1, \hat{y}_j) \geqslant e \quad \text{for almost every } j \in J'. \tag{5.10}$$

Case 2a. $J'' := \{j \in J' : d(s_j - 1, \hat{y}_j) > e\}$ is infinite.

Our assumption (5.5) on ν_1 implies that

$$\lim_s 2^{e,k}(s) < \infty \quad \text{for every } k \in N. \tag{5.11}$$

Fix k_0, $j_0 \in N$. Then by (5.11) there is a $j \in J''$ such that $j \geqslant j_0$ and $2^{e,k_0}(t_{\hat{y}_j}) < \hat{y}_j$. Since $d(s_j - 1, \hat{y}_j) > e$, there is some $\nu > \nu_2$ with $\nu \in \mathcal{M}(\hat{y}_j)$ and $D^{\nu_2,k_0}(t_{\hat{y}_j}) < \hat{y}_j$ by Rule $\hat{R}_4$. Since j_0 is arbitrary,

$$\lim_s D^{\nu_2,k_0}(s) < \infty. \tag{5.12}$$

Since k_0 was also arbitrary, (5.12) implies that infinitely many elements of $A - B$ cause $\nu_2 \in \mathcal{S}(D)$. But each of these elements causes some $\nu' \in \mathcal{S}(C)$ for some $\nu' \,{}_\sigma\!\geqslant \nu_2$ (namely at the stage immediately before they caused $\nu_2 \in \mathcal{S}(D)$.)

Case 2b. For almost every $j \in J'$, $d(s_j - 1, \hat{y}_j) = e$.

Since the claim holds for ν_3, there is according to Claim 1 a state $\nu_0 \,{}_\tau\!\geqslant \nu_3$ such that infinitely many elements of $A - B$ cause ν_0 in $\mathcal{S}(D)$. By Lemma 5.4 we can choose ν_0 so that $\nu_0 \in \mathcal{M}(\hat{x})$ for almost every $\hat{x} \in \hat{A} - \hat{B}$. Fix $k_0, j_0 \in N$. By the hypothesis on ν_0, $\lim_s D^{\nu_0,k_0}(s) < \infty$. Thus there is a $j > j_0$ such that $j \in J'$, $d(s_j - 1, \hat{y}_j) = e$, $\nu_0 \in \mathcal{M}(\hat{y}_j)$ and $D^{\nu_0,k_0}(t_{\hat{y}_j}) < \hat{y}_j$. Since $\hat{y}_j$ was put into state ν_2 at stage s_j by Rule $\hat{R}_4$, we have for k_{ν_0}, k_{ν_2} as in rule $\hat{R}_4$ $k_0 \leqslant k_{\nu_0} \leqslant k_{\nu_2}$. Further, we have that $D^{\nu_2,k_{\nu_2}}(t_{\hat{y}_j}) < \hat{y}_j$. Since j_0 is arbitrary, the Marker Lemma implies that $\lim_s D^{\nu_2,k_0}(s) < \infty$ and since k_0 was arbitrary, there are infinitely many elements of $A - B$ which cause $\nu_2 \in \mathcal{S}(D)$. This again contradicts the hypothesis on ν_2 for there is now some $\nu' \geqslant \nu_2$ such that infinitely many elements of $A - B$ cause $\nu' \in \mathcal{S}(C)$.

Now to prove the lemma we note that ν_1 occurs infinitely often in $\mathcal{T}$. Thus Claims 2 and 3 together imply that $\nu_1 \in \mathcal{P}(x)$ for almost every $\hat{x} \in \hat{A} - \hat{B}$. This contradicts our choice of ν_1 (specifically (5.5)).

(ii) is proved similarly.

Lemma 5.6. *If infinitely many elements of $\hat{A} - \hat{B}$ remain finally in pocket $\hat{Q}$ in final state ν, then infinitely many elements of $A - B$ remain finally in pocket P in final state ν.*

Proof. Assume

$$S := \{\hat{x} \in \hat{A} - \hat{B} \mid \hat{x} \text{ remain finally in } \hat{Q} \text{ in final state } \nu'\}$$

is infinite. Consider for $\hat{x} \in S$ the stage $s_{\hat{x}} + 1$ where $\hat{x}$ enters pocket $\hat{Q}$ for the last

time. Because of Rule $\hat{R}_4$ we have $\nu(s_{\hat{x}}+1, d(s_{\hat{x}}, \hat{x}), \hat{x}) \in \mathcal{M}(\hat{x})$ in case that $d(s_{\hat{x}}, \hat{x}) \geqslant 0$. Further $\nu(s_{\hat{x}}, \hat{x}, \hat{x}) = \nu(v_{\hat{x}}, \hat{x}, \hat{x})$ where $v_{\hat{x}} < s_{\hat{x}}$ is maximal such that $\hat{x}$ is on track $\hat{D}$ at the end of stage $v_{\hat{x}}$. We can then apply Lemma 5.5(i) with $\hat{X} := \hat{D}$. This implies that for every $e \in N$ there are only finitely many $x \in S$ with $d(s_{\hat{x}}, \hat{x}) < e$.

It follows from Lemma 5.1 that $\lim_s q(s, \nu'')$ exists for every $\nu'' \leqslant \nu'$. Fix some $e_0 \in N$ such that

$$(\forall \nu'' \leqslant \nu')(\exists s \geqslant e_0)[\hat{q}(s, \nu'') = \hat{q}(e_0, \nu'')].$$

We consider some state $\nu_0 \geqslant \nu'$ such that $|\nu_0| \geqslant e_0$ and $S_0 := \{\hat{x} \in S \mid \hat{x}$ has final state $\nu_0\}$ is infinite. By the preceding we have $d(s_{\hat{x}}, \hat{x}) \geqslant e_0$ for almost all $\hat{x} \in S_0$. Therefore

$$\nu_0 \leqslant \lim_t \nu(t, d(s_{\hat{x}}, \hat{x}), \hat{x}) = \nu(s_{\hat{x}}+1, d(s_{\hat{x}}, \hat{x}), \hat{x}) \in \mathcal{M}(\hat{x})$$

for almost all $\hat{x} \in S_0$. This implies that for almost all $x \in S_0$ there is some $\nu \geqslant \nu_0$ with $\nu \in \mathcal{M}(\hat{x})$ and thus $3^{\nu}(t_{\hat{x}}) < \hat{x}$.

Since $(\forall \nu \geqslant \nu_0)(\forall s \in N)[3^{\nu_0}(s) \leqslant 3^{\nu}(s)]$, this implies that $3^{\nu_0}(t_{\hat{x}}) < \hat{x}$ for almost all $\hat{x} \in S_0$. Therefore $\lim_s 3^{\nu_0}(s) < \infty$ and box $B_{\nu'}$ has a stable element by the definition of marker $3^{\nu_0}(s)$. Further every element has state ν' at the stage where it enters $B_{\nu'}$ and it doesn't change its state as long as it remains there.

If ν is as in the assumption of the lemma there are states $\nu' \geqslant \nu$ of arbitrary length such that $\{\hat{x} \in \hat{A} - \hat{B} \mid \hat{x}$ remains finally in pocket $\hat{Q}$ in final state $\nu'\}$ is infinite. Thus the claim follows from the preceding.

Lemma 5.7. *If infinitely may elements remain finally in pocket P ($\hat{P}$) in final state ν, then infinitely may elements remain finally in pocket Q ($\hat{Q}$) in final state ν.*

Proof. Assume that S in an infinite set such that every element of S remains finally in P in final state ν. For every $x \in S$ there is a state ν_x such that $\nu_x \leqslant \nu$ or $\nu \leqslant \nu_x$ and x remains finally in box B_{ν_x}. Because of Step 1 in Rule R_2 $\lim_s \hat{q}(s, \nu_x)$ exists for every $x \in S$. Further only finitely many elements remain finally in a single box. Therefore $\{\nu_x \mid x \in S\}$ is infinite. Finally for every $x \in S$ the element in $\lim_s \hat{q}(s, \nu_x)$ in $\hat{Q}$ has final state ν_x.

Lemmas 5.6 and 5.7 (and their duals) together guarantee that

$$\{x \in A - B : x \text{ has final state } \nu\} \quad \text{is infinite}$$

iff

$$\{\hat{x} \in \hat{A} - \hat{B} : \hat{x} \text{ has final state } \nu\} \quad \text{is infinite.}$$

As we argued (in Section 4) this guarantees that the desired isomorphism exists.

6. Corollaries, remarks, open questions

Corollary 6.1. *Suppose that R, $\hat{R}$ are r-maximal sets with maximal supersets. Then $\mathcal{E}^*(\bar{R}) \cong \mathcal{E}^*(\hat{R})$.*

Corollary 6.1 does not classify the automorphism type of such sets R however. For given a maximal set M, there are r.e. sets R and $\hat{R}$ such that $R \subset_{sm} M$ and $\hat{R} \subset_m M$ but not $\hat{R} \subset_s M$. R and $\hat{R}$ cannot be automorphic. The classification of automorphism types of r-maximal sets remains an important open question.

Recall that $\mathcal{M}^* = \mathcal{E}^*(A-B)$ where $B \subset_m A$. Using the theorem and existence theorems for major subsets it is now easy to see that $\mathcal{M}^*$ is a dense countable distributive lattice with each subinterval of $\mathcal{M}^*$ again isomorphic to $\mathcal{M}^*$. The countable atomless Boolean algebra is such a lattice but $\mathcal{M}^*$ is not a Boolean algebra.

The $\forall\exists$-theory of $\mathcal{M}^*$ is decidable; an important open question is the decidability of the whole theory of $\mathcal{M}^*$. Of course undecidability would yield undecidability of the theory of $\mathcal{E}^*$.

Another obvious and related open question is to characterize the structure of $\mathcal{M}^*$. Since $\mathcal{M}^*$ arises as the isomorphism type of $\mathcal{E}^*(\bar{A})$ for some r.e. sets A, this is an example of such an isomorphism type which is not a Boolean algebra. Greater understanding of these is needed to completely characterize Aut($\mathcal{E}$).

Reference

[1] E. Herrmann, The lattice of recursively enumerable sets (German), Seminarbericht Nr. 10, Humboldt University, Berlin, 1978.
[2] A.H. Lachlan, On the lattice of recursively enumerable sets, Trans. Amer. Math. Soc. 130 (1968) 1–37.
[3] A.H. Lachlan, The elementary theory of recursively enumerable sets, Duke Math. J. 35 (1968) 123–146.
[4] M. Lerman, R.A. Shore and R.I. Soare, r-maximal major subsets, Israel J. Math. 31 (1978).
[5] W. Maass, Characterization of recursively enumerable sets with supersets effectively isomorphic to all recursively enumerable sets, to appear.
[6] R.I. Soare, Automorphism of the lattice of recursively enumerable sets, Part I: Maximal sets, Ann. of Math. 100 (1974) 80–120.
[7] R.I. Soare, Automorphisms of the lattice of recursively enumerable sets, Part II: Low sets, to appear.
[8] M. Stob, Invariance of properties under automorphisms of the lattice of recursively enumerable sets, Pacific J. Math. 100 (1982) 445–471.
[9] M. Stob, Doctoral Dissertation, University of Chicago, Chicago, 1979.