

## Major Subsets and Automorphisms of Recursively Enumerable Sets

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**1. Introduction.** For recursively enumerable (r.e.) sets  $M$  and  $A$  with  $M \subseteq A$  one says that  $M$  is a major subset of  $A$  ( $M \subset_m A$ ) if  $A - M$  is infinite and if  $A \cup W = N \Rightarrow M \cup W =^* N$  for every r.e. set  $W$  ( $=^*$  means equality up to finitely many numbers,  $N$  is the set of all natural numbers).

According to Lachlan [4] every nonrecursive r.e. set  $A$  has a major subset  $M$ . An abundance of results has been proved about major subsets (see Soare [17] for a survey) since they are of critical importance for questions concerning decidability and automorphisms of the lattice of r. e. sets  $\mathcal{E}^*$ .

In Maass and Stob [13] it was shown that for any  $M \subset_m A$  and  $\tilde{M} \subset_m \tilde{A}$  there is an effective isomorphism between the intervals  $\mathcal{E}^*(A - M)$  and  $\mathcal{E}^*(\tilde{A} - \tilde{M})$ . In this paper we point out some consequences of this result and answer two related questions.

The Leitmotiv of a large part of this paper is the structural resemblance between  $\text{semilow}_{1,5}$  sets and sets in an interval bounded by a major subset. One calls an r.e. set  $D$   $\text{semilow}_{1,5}$  if there is a total recursive function  $f$  s.t., for all  $e \in N$ ,

$$W_e \cap \bar{D} \text{ infinite} \Leftrightarrow W_{f(e)} \text{ infinite.}$$

If we consider sets  $M \subset_m A$  and  $D$  with  $M \subseteq D \subset_\infty A$ , then  $D \subset_m A$  and therefore the effective isomorphism from [13] supplies a recursive function  $f$  s.t., for all  $e \in N$ ,

$$W_e \cap \bar{D} \cap (A - M) \text{ infinite} \Leftrightarrow W_{f(e)} \cap (A - M) \text{ infinite.}$$

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The outer splitting property is another common feature of  $\text{semilow}_{1,5}$  sets and sets in an interval bounded by a major subset [10, 13].

In §1 we use the splitting property from [12] in order to give a negative answer to a question of M. Lerman: Assuming  $M \subset_m A$  and  $\tilde{M} \subset_m A$ , is there an isomorphism  $\Phi: \mathcal{E}^*(A) \rightarrow \mathcal{E}^*(A)$  with  $\Phi(M^*) = \tilde{M}^*$ ?

In §2 we consider automorphisms of  $\mathcal{M}^* := \mathcal{E}^*(A - M)$  for  $M \subset_m A$ . We generalize the notions promptly simple and semilow to  $\mathcal{M}^*$  and show that promptly simple semilow sets are automorphic in  $\mathcal{M}^*$ .

In §3 we use another slight extension of the isomorphism construction from [13] in order to derive a few basic facts about the homomorphisms of Boolean algebras that are generated in  $\mathcal{E}^*$  by major subsets.

In §4 we provide the antithesis to the previously exploited lowness properties of major subsets by showing that no major subset (and thus no  $r$ -maximal set) is semi-low<sub>2</sub>.

For an arbitrary set  $S \subseteq N$  one defines

$$\mathcal{E}(S) := \{ W \cap S \mid W \text{ r.e.} \}.$$

$\mathcal{E}(S)$  is a lattice under set-theoretic union and intersection.

We write

$$\mathcal{E}_c(S) := \{ U \mid U \in \mathcal{E}(S) \text{ and } S - U \in \mathcal{E}(S) \}$$

for the sublattice of complemented elements in  $\mathcal{E}(S)$ . Obviously  $\mathcal{E}_c(S)$  is a Boolean algebra.

We write  $T^*$  for the equivalence class of  $T$  w.r.t. the equivalence relation  $=^*$ . We write  $\mathcal{E}^*(S)$ ,  $\mathcal{E}_c^*(S)$  for the corresponding quotient lattices.

CONVENTION. (1) Capital letters denote r.e. sets (unless we say “an arbitrary set”).

(2) We say that an r.e. set  $A$  is semilow,  $\text{semilow}_{1,5}$ , etc. instead of saying that the complement of  $A$  has these properties as in the original definitions in Soare [16] and Bennison and Soare [1].

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**2. The splitting property for major subsets.** M. Lerman has raised the following question. Assume that  $M$  and  $\tilde{M}$  are two major subsets of an r.e. set  $A$ . Does there exist an automorphism  $\Phi$  of  $\mathcal{E}^*(A)$  with  $\Phi(M^*) = \tilde{M}^*$ ?

It is tempting to believe that such an automorphism exists. Soare [15] has shown that any two maximal sets are automorphic in  $\mathcal{E}^*$  and the construction of a major subset of a given set  $A$  can be arranged to look very similar to the construction of a maximal set inside the universe  $A$ . In both cases,  $A - M$  consists of the final resting places of infinitely many markers  $\Gamma_e$ ,  $e \in N$ , that seek to maximalize their  $e$ -state w.r.t. certain arrays of r.e. sets (see Soare [17, Theorem 8.2]). Nevertheless, the answer to the question above is no.

**THEOREM 2.1.** *Assume  $A$  is nonrecursive. Then there are major subsets  $M, \tilde{M}$  of  $A$  such that  $\Phi(M^*) \neq \tilde{M}^*$  for every automorphism  $\Phi$  of  $\mathcal{E}^*(A)$ .*

In order to prove this theorem we consider the following property.

**DEFINITION 2.2.** *Assume  $B \subseteq D \subseteq A$ . We say that  $D$  has the splitting property in  $A - B$  if one can split every r.e. set  $W \subseteq A$  into r.e. sets  $W', W''$  s.t.  $W' \subseteq D$  and*

$$(A - W) \cup B \text{ not r.e.} \Rightarrow (A - W') \cup B, (A - W'') \cup B \text{ not r.e.}$$

If we take  $A := N$  and  $B := \emptyset$  this definition coincides with the weak splitting property from Maass, Shore and Stob [12].

Notice that one can define the splitting property in  $A - B$  by an elementary definition over  $\mathcal{E}^*(A - B)$ . We show in Lemma 2.3 that every nonrecursive set  $A$  has a major subset  $M$  that has the splitting property in  $A$ , and in Lemma 2.5 that every nonrecursive set  $A$  has, as well, a major subset  $\tilde{M}$  that has not the splitting property in  $A$ . This will finish the proof of Theorem 2.1. Observe that this argument exploits the structural resemblance to low sets, where the splitting property can be used to show that not all low sets are automorphic in  $\mathcal{E}^*$ .

**LEMMA 2.3.** *Assume  $A$  is not recursive. Then there is a major subset  $M \subset_m A$  s.t.  $M$  is promptly simple in  $A$  and therefore has the splitting property in  $A$ .*

**PROOF.** A standard construction of a major subset  $M \subset_m A$  as e.g. in Soare [17, Theorem 8.2] can obviously be combined with the satisfaction of the standard finitary positive requirements that make  $M$  promptly simple in  $A$ . According to Theorem 2.2 in [12] prompt simplicity implies the splitting property and this still holds if we substitute the universe  $N$  by  $A$ .

**LEMMA 2.4.** *Assume  $B \subseteq D \subseteq A$ ,  $\mathcal{E}^*(A - D)$  is not a Boolean algebra and the Turing degree of  $D$  is half of a minimal pair. Then  $D$  does not have the splitting property in  $A - B$ .*

**PROOF.** This is shown in Theorem 3.1 in Maass, Shore and Stob [12] for the case  $A = N$  and  $B = \emptyset$ . The same argument works as well in this more general situation.

**LEMMA 2.5.** *Assume  $A$  is not recursive. Then there is a major subset  $M \subset_m A$  s.t.  $M$  has not the splitting property in  $A$ .*

**PROOF.** Let  $\mathbf{h}$  be a high Turing degree that is half of a minimal pair (see Lachlan [2]). Since  $\mathbf{h}$  is high there exists a major subset  $M \subset_m A$  of degree  $\mathbf{h}$  (Lerman [5]). We then apply Lemma 2.4 with  $B := \emptyset$  and  $D := M$ , and see that  $M$  does not have the splitting property in  $A$ .

One could as well construct directly a set  $M$  with the desired properties.

**REMARK 2.6.** *Assume  $M \subset_m A$ . One can use the splitting property in  $A - M$  in order to show, for various properties  $P$ , that not all sets  $D$  in  $\mathcal{E}(A - M)$  with property  $P$  are automorphic in  $\mathcal{E}^*(A - M) =: \mathcal{M}^*$ . For many  $P$  it is easy to construct a set  $D$  with property  $P$  that has, in addition, the splitting property in*

$A - M$ . If there exist, in addition, sets  $D$  with property  $P$  in every high degree, one can use Lemma 2.4 with  $B := M$  in order to get some  $D$  with property  $P$  that does not have the splitting property in  $A - M$ .

**3. Automorphisms of  $\mathcal{M}^*$ .** It is not so easy to find sets that are automorphic in  $\mathcal{M}^*$ , since there are no sets with few supersets in  $\mathcal{M}^*$  (like e.g. maximal sets in  $\mathcal{E}^*$ ). We show here that there are analogies of promptly simple and semilow sets in  $\mathcal{M}^*$  and that sets with both properties are automorphic in  $\mathcal{M}^*$ .

Since it is easy to make sets in  $\mathcal{M}^*$  promptly simple and semilow, and since all these sets realize the same 1-type in  $\mathcal{M}^*$ , we suggest as the next step to look for a decision procedure for  $\forall\exists$ -sentences in  $\mathcal{M}^*$ , which have a promptly simple semilow set in  $\mathcal{M}^*$  as parameter (a decision procedure for  $\forall\exists$ -sentences without parameters in  $\mathcal{M}^*$  is given in Stob [18]). This will supply valuable experience towards a decision procedure for the  $\exists\forall\exists$ -theory of  $\mathcal{M}^*$ .

**DEFINITION 3.1.** Consider sets  $B, D, A$  with  $B \subseteq D \subseteq A$ .

(a) We say that  $D$  is *promptly simple* in  $A - B$  if  $A - D$  is infinite and if there is a recursive function  $f$  and an enumeration of all r.e. subsets  $(U_i)_{i \in \mathbb{N}}$  of  $A$  s.t.

$$\forall j \in \mathbb{N} (U_j \cap (A - B) \text{ infinite} \Rightarrow \exists x, s (x \in U_{j,s} - U_{j,s-1} \\ \wedge x \notin B \wedge x \in D_{f(s)})).$$

(b) We say that  $D$  is *semilow* in  $A - B$  if there is an enumeration of  $D$  and all r.e. subsets  $(U_i)_{i \in \mathbb{N}}$  of  $A$  s.t.

$$\forall i ((U_i \setminus D) \cap (A - B) \text{ infinite} \Rightarrow (U_i - D) \cap (A - B) \text{ infinite}).$$

**REMARK 3.2.** Because of Theorem 5.1 below not every characterization of semilow can be generalized to an interval  $A - M$  with  $M \subset_m A$ . Of course the preceding definition is equivalent to the standard definition of semilow if  $A - B = \mathbb{N}$ . Note that sets in  $\mathcal{E}_C(A - B)$  are always semilow in  $A - B$ .

**THEOREM 3.3.** Assume  $M \subset_m A$ . Then there is a set  $D$  with  $M \subseteq D \subseteq A$  s.t.  $D$  is promptly simple in  $A - M$  and  $D$  is semilow in  $A - M$ .

**PROOF.** We use similar arguments as in Stob [18].

We construct a recursive enumeration of a set  $D$  s.t.  $M \subseteq D \subseteq A$  and s.t. with respect to the standard enumeration of  $(W_e)_{e \in \mathbb{N}}$  the following requirements are satisfied for all  $k, i, n, j \in \mathbb{N}$ :

$$N_k: |A - D| \geq k,$$

$$N_{i,n}: (W_i \setminus D) \cap (A - M) \text{ infinite} \Rightarrow |(W_i - D) \cap (A - M)| \geq n$$

and

$$P_j: W_j \cap (A - M) \text{ infinite} \Rightarrow \exists x, s (x \in W_{j,s} \cap (A_s - W_{j,s-1}) \cap A_{s-1} \\ \wedge x \notin M \wedge x \in D_s).$$

We define in the usual fashion (see [13]) nondecreasing recursive functions ('movable markers')  $\Gamma_{N_k}(s), \Gamma_{N_{i,n}}(s), \Gamma_{P_j}(s)$  s.t.

$$\lim_{s \rightarrow \infty} \Gamma_{N_k}(s) = \infty \quad \text{iff} \quad |A - D| < k,$$

$$\lim_{s \rightarrow \infty} \Gamma_{N_{i,n}}(s) = \infty \quad \text{iff} \quad \text{the conclusion of } N_{i,n} \text{ does not hold}$$

and

$$\lim_{s \rightarrow \infty} \Gamma_{P_j}(s) = \infty \quad \text{iff} \quad \text{the conclusion of } P_j \text{ does not hold.}$$

THE CONSTRUCTION. We fix enumerations of  $A$  and  $M$ . For  $x \in A$  we define  $s_x := \mu s(x \in A_s)$ .

Stage  $s$ . We consider all  $x \in A_s$ . We place  $x$  in  $D$  if  $x \in M_s$  or if there exists some  $j \in N$  s.t.

$$\begin{aligned} x &\in W_{j,s} \cap (A_s - W_{j,s-1}) \cap A_{s-1}, \\ \Gamma_{P_j}(s_x) &> x, \\ \Gamma_{N_k}(s_x) &\leq x \quad \text{for all } k \leq j, s \\ \Gamma_{N_{i,n}}(s_x) &\leq x \quad \text{for all } \langle i, n \rangle \leq j \text{ with } x \in W_{i,s-1} \end{aligned}$$

(in this case we say that  $P_j$  forces  $x$  into  $D$ ).

LEMMA 3.4. *Every requirement  $P_j$  forces only finitely many elements of  $A - M$  into  $D$ .*

PROOF. If  $P_j$  forces some element of  $(A - M)$  into  $D$  then  $\lim_{s \rightarrow \infty} \Gamma_{P_j}(s) < \infty$ .

LEMMA 3.5. *Every requirement  $N_k$  is satisfied.*

PROOF. Otherwise,  $\Gamma_{N_k}(s_x) > x$  for almost all  $x \in A - M$ .

LEMMA 3.6. *Every requirement  $N_{i,n}$  is satisfied.*

PROOF. Assume  $N_{i,n}$  is not satisfied. Thus  $(W_i \setminus D) \cap (A - M)$  is infinite and  $\lim_{s \rightarrow \infty} \Gamma_{N_{i,n}}(s) = \infty$ . Since  $M \subset_m A$  we have  $\Gamma_{N_{i,n}}(s_x) > x$  for almost all  $x \in A - M$ . Therefore, almost all of the infinitely many elements of  $(W_i \setminus D) \cap (A - M)$  can only be forced into  $D$  by  $P_j$  with  $j < \langle i, n \rangle$ . Thus  $N_{i,n}$  is satisfied since these  $P_j$  together force only finitely many elements of  $A - M$  into  $D$  according to Lemma 3.4, a contradiction.

LEMMA 3.7. *Every requirement  $P_j$  is satisfied.*

PROOF. This follows easily from Lemmas 3.5 and 3.6.

This finishes the proof of Theorem 3.3.

THEOREM 3.8. *Assume  $M \subset_m A$  and  $\tilde{M} \subset_m \tilde{A}$ . Further assume that  $M \subseteq D \subseteq A$ ,  $\tilde{M} \subseteq \tilde{D} \subseteq \tilde{A}$  and  $D, \tilde{D}$  are promptly simple and semilow in  $A - M$ , resp.  $\tilde{A} - \tilde{M}$ . Then there is an isomorphism*

$$\Phi: \mathcal{E}^*(A - M) \rightarrow \mathcal{E}^*(\tilde{A} - \tilde{M}) \quad \text{with } \Phi(D^*) = \tilde{D}^*.$$

PROOF. First we notice that  $D \subset_m A$  and  $\tilde{D} \subset_m \tilde{A}$ . Therefore there exists an effective isomorphism [13]

$$\psi: \mathcal{E}^*(A - D) \rightarrow \mathcal{E}^*(\tilde{A} - \tilde{D}).$$

More exactly, there are simultaneously recursively enumerable arrays  $(U_i)_{i \in N}$ ,  $(\hat{V}_i)_{i \in N}$ ,  $(\hat{U}_i)_{i \in N}$  and  $(V_i)_{i \in N}$  s.t., for every  $i \in N$ ,  $U_i = *W_i = *V_i$  and s.t., for every state  $\nu$  (see [10] for notation), infinitely many elements of  $A - D$  have state  $\nu$  w.r.t.  $(U_i)_{i \in N}$ ,  $(\hat{V}_i)_{i \in N}$  iff infinitely many elements of  $\tilde{A} - \tilde{D}$  have state  $\nu$  w.r.t.  $(\hat{U}_i)_{i \in N}$ ,  $(V_i)_{i \in N}$ .

It follows from Lemma 5.5 in [13] that, in addition, the following weak covering property (\*) holds (notation as in [13]).

(\*) For every state  $\nu$ : if infinitely many elements of  $A - D$  have state  $\nu$  w.r.t.  $(U_i)_{i \in N}$ ,  $(\hat{V}_i)_{i \in N}$  at some point of the enumeration then there is a state  $\nu_1 \leq \nu$  s.t. infinitely many elements of  $\tilde{A} - \tilde{D}$  have state  $\nu_1$  w.r.t.  $(\hat{U}_i)_{i \in N}$ ,  $(V_i)_{i \in N}$  at some point of the enumeration + symmetrical counterpart.

We proceed then analogously as for low sets in Maass [11]. We relativize the prompt and low shrinking property of [11] to the interval  $A - M$ . Since  $D$  is promptly simple and semilow in  $A - M$ ,  $D$  has the prompt and low shrinking property in  $A - M$  (this is proved exactly as in the unrelativized case [11]). Analogously  $\tilde{D}$  has the prompt and low shrinking property in  $\tilde{A} - \tilde{M}$ . This implies via a relativized version of the Shrinking Lemma in [11] that we can shrink the sets  $\hat{V}_i$  to sets  $V'_i \subseteq \hat{V}_i$  and the sets  $\hat{U}_i$  to sets  $U'_i \subseteq \hat{U}_i$  s.t.

$$V'_i \cap (A - D) = * \hat{V}_i \cap (A - D), \quad U'_i \cap (\tilde{A} - \tilde{D}) = * \hat{U}_i \cap (\tilde{A} - \tilde{D}),$$

and s.t., in addition, the covering property (\*\*) of the following Extension Theorem for  $\mathcal{M}$  is satisfied. The conclusion of the Extension Theorem for  $\mathcal{M}$  implies that we can continue the isomorphism  $\psi: \mathcal{E}^*(A - D) \rightarrow \mathcal{E}^*(\tilde{A} - \tilde{D})$  to an automorphism  $\Phi: \mathcal{E}^*(A - M) \rightarrow \mathcal{E}^*(\tilde{A} - \tilde{M})$  with  $\Phi(D^*) = \tilde{D}^*$ .

THEOREM 3.9 (EXTENSION THEOREM FOR  $\mathcal{M}$ ). Assume  $M \subset_m D$  and  $\tilde{M} \subset_m \tilde{D}$ . Further assume that there is a simultaneous enumeration of  $M$ ,  $D$ ,  $\tilde{M}$ ,  $\tilde{D}$  and arrays

$$(U_i)_{i \in N}, (V'_i)_{i \in N}, (U'_i)_{i \in N}, (V_i)_{i \in N} \text{ s.t. for every } i \in N \\ D \searrow V'_i = \emptyset \quad \text{and} \quad \tilde{D} \searrow U'_i = \emptyset,$$

and s.t. the following covering property (\*\*) holds.

(\*\*) For every state  $\nu$ : if infinitely many elements of  $D - M$  enter in state  $\nu$  w.r.t.  $(U_i)_{i \in N}$ ,  $(V'_i)_{i \in N}$  then there is a state  $\nu_1 \leq \nu$  s.t. infinitely many elements of  $\tilde{D} - \tilde{M}$  enter  $\tilde{D}$  in state  $\nu_1$  w.r.t.  $(U'_i)_{i \in N}$ ,  $(V_i)_{i \in N}$  + symmetrical counterpart.

Then one can extend the sets  $V'_i$ ,  $U'_i$  to sets  $\tilde{V}_i$ ,  $\tilde{U}_i$  s.t., for every  $i \in N$ ,

$$\tilde{V}_i \cap \bar{D} = V'_i \cap \bar{D}, \quad \tilde{U}_i \cap \bar{\tilde{D}} = U'_i \cap \bar{\tilde{D}},$$

and s.t. for every state  $\nu$  infinitely many elements of  $D - M$  have state  $\nu$  w.r.t.  $(U_i)_{i \in \mathbb{N}}, (\tilde{V}_i)_{i \in \mathbb{N}}$  iff infinitely many elements of  $\tilde{D} - \tilde{M}$  have state  $\nu$  w.r.t.  $(\tilde{U}_i)_{i \in \mathbb{N}}, (\tilde{V}_i)_{i \in \mathbb{N}}$ .

PROOF. The construction is a trivial extension of the construction in [13]. The only difference in the proof occurs at the beginning of the verification of Claim 3 in the proof of [13, Lemma 5.5]. At this point one uses covering property (\*\*) instead of the argument: "Of course  $\sigma_2 \neq \emptyset \dots$ ".

**4. On the structure of subalgebras and homomorphisms of Boolean algebras generated by major subsets.** If  $S, \tilde{S}$  are two arbitrary infinite sets with  $S \subset \tilde{S}$ , then the Boolean algebra  $B'$  of intersection with  $S$  of r.e. sets that look recursive on  $\tilde{S}$  (i.e.  $\{(R \cap S)^* | R^* \in \mathcal{E}^*(\tilde{S}) \text{ and } (\tilde{S} - R)^* \in \mathcal{E}^*(\tilde{S})\}$ ) is a subalgebra of the Boolean algebra  $B$  of r.e. sets that look recursive on  $S$  (i.e.  $\{U^* | U^* \in \mathcal{E}^*(S) \text{ and } (S - U)^* \in \mathcal{E}^*(S)\} =: \mathcal{E}_C^*(S)$ ). In general when  $\tilde{S}$  grows, fewer sets look recursive on  $\tilde{S}$  and thus the subalgebra  $B' \hookrightarrow B$  shrinks.

We consider here only the case where  $\tilde{S} := N$  (thus the sets that look recursive on  $\tilde{S}$  are the real recursive sets) and  $S := A - M$  for r.e. sets  $A$  and  $M$ .

Discussions with M. Stob led to the observation that the Owings Splitting Theorem restricts the subalgebras  $B' \hookrightarrow B$  that arise in this way: if  $B' \hookrightarrow B$  is represented as above then one can split every element of  $B - B'$  into two elements of  $B - B'$ . It is not difficult to see that there are embeddings of Boolean algebras into the countable atomless Boolean algebra which do not have this special property.

The characterization of all subalgebras of Boolean algebras that arise in  $\mathcal{E}^*$  in this way is an interesting although quite difficult project.

We restrict our attention here to those subalgebras that arise in the case where  $M \subset_m A$ . It is well known that the subalgebra  $B' \hookrightarrow B$  that is represented by  $\tilde{S} := N$  and  $S := A - M$  (with  $M, A$  r.e.) as above satisfies

$$\forall b \in B (b \neq 0 \Rightarrow \exists a \in B (a < b \wedge a \notin B'))$$

iff  $M \subset_m A$  (this follows as well from Corollary 4.5). In particular,  $B$  is always the atomless countable Boolean algebra if  $M \subset_m A$ .

In the case  $M \subset_m A$  we have the special property that every recursive set  $R$  has (up to  $=^*$ ) a unique continuation from  $\bar{A}$  onto  $\bar{M}$ . Thus the function

$$H_{A,M}: \mathcal{E}_C^*(\bar{A}) \rightarrow \mathcal{E}_C^*(A - M)$$

defined by  $(R \cap \bar{A})^* \mapsto (R \cap (A - M))^*$  for recursive sets  $R$  is independent of the choice of  $R$  and therefore well defined. Obviously  $H_{A,M}$  is a homomorphism of Boolean algebras where  $\text{Range}(H_{A,M})$  is the previously considered subalgebra  $B'$ . By choosing suitable  $A$  we thus get some control over the subalgebra  $B'$ . Observe that  $H_{A,M}$  is 1-1 if  $M$  is a small major subset of  $A$  and  $\text{Range}(H_{A,M})$  is just the 0-1 Boolean algebra if  $M$  is an  $r$ -maximal major subset of  $A$ .

These maps  $H_{A,M}$  are of interest with regard to the characterization of orbits of r.e. sets under automorphisms of  $\mathcal{E}^*$ . The characterization of the orbit, in which an r.e. set  $A$  lies, requires a full understanding of the function

$$\mathcal{E}^*(\bar{A}) \ni (W \cap \bar{A})^* \mapsto \{X^* \mid X \subseteq A, X \text{ r.e. and } X \cup (W - A) \text{ r.e.}\} \subseteq \mathcal{E}^*(A).$$

The preceding homomorphisms  $H_{A,M}$  are embryos of these functions, where we consider only recursive sets  $W$  and restrict our attention to the smaller lattice  $\mathcal{E}^*(A - M)$  (instead of  $\mathcal{E}^*(A)$ ) where the continuation of the recursive set  $W$  is still unique.

**THEOREM 4.1.** *Assume  $M \subset_m A$ ,  $\tilde{M} \subset_m \tilde{A}$  and, for every recursive set  $R$ ,*

$$R \cap (A - M) =^* \emptyset \Leftrightarrow R \cap (\tilde{A} - \tilde{M}) =^* \emptyset.$$

*Then there is an isomorphism*

$$\Phi: \mathcal{E}^*(A - M) \rightarrow \mathcal{E}^*(\tilde{A} - \tilde{M})$$

*s.t., for every recursive set  $R$ ,*

$$\Phi((R \cap (A - M))^*) = (R \cap (\tilde{A} - \tilde{M}))^*.$$

**PROOF.** This is another inessential extension of the isomorphism construction in Maass and Stob [13]. We fix a simultaneous enumeration of an array  $(R_i)_{i \in \mathbb{N}}$  that contains exactly the recursive sets: at Stage  $s$  we enumerate  $x$  in  $R_i$  if

$$\phi_{i,s}(x) \simeq 1 \quad \text{and} \quad \forall y < x (\phi_{i,s}(y) \downarrow \text{ and } \phi_{i,s}(y) \in \{0, 1\}).$$

**LEMMA 4.2.** *Assume  $e \in \mathbb{N}$ . Almost all  $x \in A - M$  have the property*

$$\{i \leq e \mid x \in R_i\} = \{i \leq e \mid x \in R_{i,t_x}, \text{ where } t_x := \mu t (x \in A_t)\}.$$

*The same holds for  $\tilde{A} - \tilde{M}$ .*

**PROOF.** If  $R_i \cap (A - M)$  is infinite, then  $x \in R_{i,t_x}$  for almost all  $x \in R_i \cap (A - M)$ .

We need an extension of the construction in Maass and Stob [13] similar to the one that was needed for the proof of the Extension Theorem for  $\mathcal{M}$ . We have to take into account that elements are already in certain sets  $R_i$  before they appear in the construction. But unlike the situation in the Extension Theorem, we do not even have to extend these  $R_i$  to get an exact matching of states on both sides. Lemma 4.2 enables us to perform the construction of [13] simultaneously but separately inside every  $e$ -state of the  $R_i$ .

States  $\nu$  are now 4-tuples  $\langle e, \sigma, \tau, \rho \rangle$  (instead of tuples  $\langle e, \sigma, \tau \rangle$ ) as in [13] with  $\sigma, \tau, \rho \subseteq e + 1$ .

For the construction in  $A - M$  we say that  $x \in A$  has state  $\langle e, \sigma, \tau, \rho \rangle$  at Stage  $s$  if

$$\sigma = \{i \leq e \mid x \in U_{i,s}\}, \quad \tau = \{i \leq e \mid x \in \hat{V}_{i,s}\}$$

and

$$\rho = \{i \leq e \mid x \in R_{i,t_x}\},$$



where  $t_x$  is the stage where the simultaneous enumeration function  $g$  enumerated  $x$  into  $A$ .

For the construction in  $\tilde{A} - \tilde{M}$  we say that  $\tilde{x} \in \tilde{A}$  has state  $\langle e, \sigma, \tau, \rho \rangle$  at Stage  $s$  if

$$\sigma = \{i \leq e \mid x \in \hat{U}_{i,s}\}, \quad \tau = \{i \leq e \mid x \in V_{i,s}\}$$

and

$$\rho = \{i \leq e \mid x \in R_{i,t_x}\},$$

where  $t_{\tilde{x}}$  is the stage where  $\tilde{x}$  entered  $\tilde{A}$ .

For  $\nu = \langle e, \sigma, \tau, \rho \rangle$  and  $\nu' = \langle e', \sigma', \tau', \rho' \rangle$  we say that  $\nu \leq \nu'$  ( $\nu'$  extends  $\nu$ ) if  $e \leq e', \sigma = \sigma' \cap (e + 1), \tau = \tau' \cap (e + 1)$  and  $\rho = \rho' \cap (e + 1)$ .

We say that  $\nu \geq \nu'$  ( $\nu$  covers  $\nu'$ ) if  $e = e', \sigma \supseteq \sigma', \tau \supseteq \tau'$  and  $\rho = \rho'$ .

The goal of the construction is to get sets  $U_i, \hat{V}_i, \hat{U}_i, V_i$  s.t., for every  $i \in N$ ,

$$U_i = {}^*W_i \cap (A - M) \quad \text{and} \quad V_i = {}^*W_i \cap (\tilde{A} - \tilde{M})$$

and s.t. for every state  $\nu = \langle e, \sigma, \tau, \rho \rangle$  infinitely many  $x \in A - M$  have final state  $\nu$  iff infinitely many  $\tilde{x} \in \tilde{A} - \tilde{M}$  have final state  $\nu$ .

Of course, a stream  $\mathcal{S}(X)$  consists now of 4-tuples  $\nu = \langle e, \sigma, \tau, \rho \rangle$  and boxes  $B_\nu$ , a well as the function  $q(\mathcal{S}, \nu)$ , are now defined for these extended states  $\nu$ .

The only change in the proof of [13] occurs in the verification of Claim 3 in the proof of [13, Lemma 5.5]. One argues now as follows.

By contradiction fix  $\nu_2 = \langle e, \sigma_2, \tau_2, \rho_2 \rangle$  such that the claim fails for  $\nu_2$ ,  $\sigma_2$  is minimal and  $\tau_2$  is minimal for  $\sigma_2$ . Assume first that  $\sigma_2 = \emptyset$ . Since  $\nu_2$  occurs infinitely often in  $\mathcal{S}$ , there are infinitely many  $\hat{y} \in \tilde{A} - \tilde{M}$  s.t.

$$\rho_2 = \{i \leq e \mid \hat{y} \in R_{i,t_{\hat{y}}}\}.$$

By the preceding lemma this implies, for the recursive set  $R := \bigcap \{R_i \mid i \in \rho_2\}$ , that  $R \cap (\tilde{A} - \tilde{M})$  is infinite. By the assumption of the theorem this implies that  $R \cap (A - M)$  is as well infinite. Thus, by Lemma 4.2,

$$S := \{x \in A - M \mid \{i \leq e \mid x \in R_{i,t_x}\} = \rho_2\}$$

is infinite. All  $x \in S$  are in some state  $\nu' = \langle e, \sigma', \tau', \rho_2 \rangle$  with  $\tau' = \emptyset$  when they run for the first time over track  $\mathcal{G}$ . We have  $\nu' \geq \nu$  for such a state  $\nu'$  and therefore Claim 3 holds for  $\nu_2$ , a contradiction. One argues then for  $\sigma_2 \neq \emptyset$  as in [13]. The analogous versions of Lemmas 5.6 and 5.7 from [13] show that the construction meets our previously mentioned goal.

This finishes the proof of Theorem 4.1.

**COROLLARY 4.3.** *Assume  $M$  and  $\tilde{M}$  are major subsets of  $A$  s.t. the maps*

$$H_{A,M}: \mathcal{E}_C^*(\bar{A}) \rightarrow \mathcal{E}_C^*(A - \tilde{M}) \quad \text{and} \quad H_{A,\tilde{M}}: \mathcal{E}_C^*(\bar{A}) \rightarrow \mathcal{E}_C^*(A - \tilde{M})$$

*have the same kernel. Then  $H_{A,M}$  and  $H_{A,\tilde{M}}$  are identical up to isomorphism, i.e. there exists an isomorphism  $\Phi: \mathcal{E}^*(A - M) \rightarrow \mathcal{E}^*(A - \tilde{M})$  s.t., for all recursive  $R$ ,*

$$H_{A,\tilde{M}}((R \cap \bar{A})^*) = \Phi(H_{A,M}((R \cap \bar{A})^*)).$$

PROOF. This follows immediately from Theorem 4.1 and the definition of  $H_{A,M}$  before Theorem 4.1.

REMARK 4.4. The previous corollary shows that for a fixed  $A$  the homomorphisms  $H_{A,M}$  for  $M \subset_m A$  are already determined by their kernel. This becomes different if we let  $\bar{A}$  vary. The following is an example of sets  $A, \tilde{A}$ , where both  $\mathcal{E}^*(\bar{A})$  and  $\mathcal{E}^*(\tilde{A})$  are the countable atomless Boolean algebra, and of sets  $M \subset_m A$  and  $\tilde{M} \subset_m \tilde{A}$  s.t. both  $H_{A,M}$  and  $H_{\tilde{A},\tilde{M}}$  are 1-1 embeddings of the countable atomless Boolean algebra into the countable atomless Boolean algebra  $\mathcal{E}_C^*(A - M)$ , resp.  $\mathcal{E}_C^*(\tilde{A} - \tilde{M})$ , but where the Boolean algebra  $\mathcal{E}_C^*(A - M)$  together with the distinguished subalgebra  $\text{Range}(H_{A,M})$  has a different structure than  $\mathcal{E}_C^*(\tilde{A} - \tilde{M})$  together with the subalgebra  $\text{Range}(H_{\tilde{A},\tilde{M}})$ . We take both  $A$  and  $\tilde{A}$  to be atomless hyperhypersimple. We choose  $A$  semilow<sub>2</sub> and  $\tilde{A}$  s.t. it does not have an  $r$ -maximal major subset (see Lerman, Shore and Soare [6]). We let  $M$  ( $\tilde{M}$ ) be any small major subset of  $A$  ( $\tilde{A}$ ). This implies immediately that both  $H_{A,M}$  and  $H_{\tilde{A},\tilde{M}}$  are 1-1 (thus they have the same kernel). Further, one can split every nonzero element of  $\mathcal{E}_C^*(\tilde{A} - \tilde{M})$  by an element in  $\text{Range}(H_{\tilde{A},\tilde{M}})$  (i.e. by a recursive set) whereas there is a nonzero element in  $\mathcal{E}_C^*(A - M)$  which cannot be split by elements in  $\text{Range}(H_{A,M})$  (actually below every nonzero element in  $\mathcal{E}_C^*(A - M)$  there is one with this property).

The following corollary shows that for  $M \subset_m A$  the homomorphism  $H_{A,M}$  has a characteristic homogeneity property.

COROLLARY 4.5. *Assume  $M \subset_m A$ . Then the homomorphism*

$$H_{A,M}: \mathcal{E}_C^*(\bar{A}) \rightarrow \mathcal{E}_C^*(A - M)$$

*has the following homogeneity property: one can split every nonzero element  $T^* \in \mathcal{E}_C^*(A - M)$  into two pieces  $T_1^*, T_2^* \in \mathcal{E}_C^*(A - M) - \text{Range}(H_{A,M})$  s.t., for every  $i \in \{1, 2\}$ , the homomorphism*

$$H_{A,M}^{T_i}: \mathcal{E}_C^*(\bar{A}) \rightarrow \mathcal{E}_C^*(T_i)$$

*with  $H_{A,M}^{T_i}(b) = H_{A,M}(b) \cap T_i^*$  for  $b \in \mathcal{E}_C^*(\bar{A})$  has the same structure as the homomorphism*

$$H_{A,M}^T: \mathcal{E}_C^*(\bar{A}) \rightarrow \mathcal{E}_C^*(T)$$

*with  $H_{A,M}^T(b) = H_{A,M}(b) \cap T^*$  for  $b \in \mathcal{E}_C^*(\bar{A})$ , i.e. there is an isomorphism*

$$\Phi_i: \mathcal{E}_C^*(T_i) \rightarrow \mathcal{E}_C^*(T)$$

*s.t., for all  $b \in \mathcal{E}_C^*(\bar{A})$ ,*

$$H_{A,M}^T(b) = \Phi_i(H_{A,M}^{T_i}(b)).$$

PROOF. Every nonzero element  $T^* \in \mathcal{E}_C^*(A - M)$  has the form  $(W - M)^*$  for some r.e.  $W$  with  $M \subseteq W \subseteq A$  and  $M \cup \bar{W}$  not r.e. According to the Owings Splitting Theorem [17] we can split  $W$  into r.e. sets  $W_1, W_2$  such that, for all r.e.  $U$  and  $i \in \{1, 2\}$ ,

$$(U - W_i) \cup M \text{ r.e.} \Rightarrow (U - W) \cup M \text{ r.e.}$$

We define  $T_i := W_i - M$ .

We show that for every recursive  $R$  we have, for  $i \in \{1, 2\}$ ,

$$(*) \quad R \cap (W - M) \text{ infinite} \Leftrightarrow R \cap (W_i - M) \text{ infinite.}$$

Assume  $R$  is recursive and  $R \cap (W_i - M)$  is finite. Then  $(R - W_i) \cup M$  is r.e. and thus  $(R - W) \cup M$  is as well r.e. The union of the r.e. sets  $(R - W) \cup M$  and  $\overline{R} \cup W$  is equal to  $N$ . We apply reduction to these sets and get a recursive  $R_0$  with  $R_0 \cap \overline{W} = R \cap \overline{W}$  and  $R_0 \cap (W - M) = \emptyset$ . Since  $M \subset_m W$ , this implies that  $R \cap (W - M)$  is finite.

We get  $T_i^* \notin \text{Range}(H_{A,M})$  directly from (\*). Further with (\*) we get the desired isomorphism  $\Phi_i$  from Theorem 4.1.

**5. Major subsets are not semilow<sub>2</sub>.** An r.e. set  $B$  is called semilow<sub>2</sub> if  $\{e | W_e \cap \overline{B} \text{ infinite}\} \leq_T 0''$ . Many classes of r.e. sets (e.g. atomless hyperhypersimple sets) that consist only of sets of high degree contain some particularly well-behaved representatives that are semilow<sub>2</sub>. These often have interesting special properties (e.g. all semilow<sub>2</sub> atomless hyperhypersimple sets are automorphic [9]). We show below that the class of major subsets (and thus the class of  $r$ -maximal sets that are not maximal) does not contain semilow<sub>2</sub> sets. On the contrary, all major subsets are as far away from being semilow<sub>2</sub> as possible. It follows from [13] that the Turing degree of the set

$$\{e | W_e \cap (A - M) \text{ infinite}\}$$

does not depend on the choice of  $A, M$  with  $M \subset_m A$ . We show here that this degree is equal to  $0'''$ . It is obvious that  $\{e | W_e \cap (A - M) \text{ infinite}\}$  is recursive in  $\{e | W_e \cap \overline{M} \text{ infinite}\}$ .

**THEOREM 5.1.** *Assume  $M \subset_m A$ . Then*

$$\text{deg}\{e | W_e \cap (A - M) \text{ infinite}\} = 0'''.$$

**PROOF.** Let  $S_3 \in 0'''$  be a  $\Sigma_3^0$  set. It is easy to construct an r.e. set  $W_{e_0}$  and a recursive function  $p$  s.t.

$$\forall e \in N (e \in S_3 \Leftrightarrow W_{p(e)} - W_{e_0} \text{ finite}).$$

If  $M \subset_m A$  then  $M$  has the outer splitting property in  $A$  [13]. Thus there is a recursive function  $g$  s.t. the sets  $W_{g(i)}$  are pairwise disjoint,

$$A = \bigcup_{i \in N} W_{g(i)}$$

and

$$\forall i \in N (W_{g(i)} \cap (A - M) \text{ is finite and nonempty}).$$

We use this to embed  $\mathcal{E}^*$  effectively into  $\mathcal{E}^*(A - M)$ . There is a recursive function  $k$  s.t., for all  $e, e' \in N$ ,

$$W_e - W_{e'} \text{ finite} \Leftrightarrow (W_{k(e)} - W_{k(e')}) \cap (A - M) \text{ finite}$$

(set  $W_{k(e)} := \bigcup_{i \in W_e} W_{g(i)}$ ).

We define  $\tilde{M} := W_{k(e_0)} \cup M$ . Then, for all  $e \in N$ ,

$$\begin{aligned} e \in S_3 &\Leftrightarrow W_{p(e)} - W_{e_0} \text{ finite} \Leftrightarrow (W_{k(p(e))} - W_{k(e_0)}) \cap (A - M) \text{ finite} \\ &\Leftrightarrow W_{k(p(e))} \cap (A - \tilde{M}) \text{ finite.} \end{aligned}$$

Thus  $S_3$  is recursive in  $\{e | W_e \cap (A - \tilde{M}) \text{ finite}\}$  and since  $\tilde{M} \subset_m A$  this set has the same degree as

$$\{e | W_e \cap (A - M) \text{ finite}\}.$$

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