THE JOURNAL OF SYMBOLIC LOGIC
Volume 43, Number 2, June 1978

THE UNIFORM REGULAR SET THEOREM IN α-RECURSION THEORY[1]

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Several new features arise in the generalization of recursion theory on ω to recursion theory on admissible ordinals α, thus making α-recursion theory an interesting theory. One of these is the appearance of irregular sets. A subset A of α is called regular (over α), if we have for all $\beta < \alpha$ that $A \cap B \in L_\alpha$, otherwise A is called irregular (over α). So in the special case of ordinary recursion theory ($\alpha = \omega$) every subset of α is regular, but if α is not a cardinal of L we find constructible sets $A \subseteq \alpha$ which are irregular. The notion of regularity becomes essential, if we deal with α-recursively enumerable (α-r.e.) sets in priority constructions (α-r.e. is defined as Σ_1 over L_α). The typical situation occurring there is that an α-r.e. set A is enumerated during some construction in which one tries to satisfy certain requirements. Often this construction succeeds only if we can insure that every initial segment $A \cap \beta$ of A is completely enumerated at some stage before α. This calls for making sure that A is regular because due to the admissibility of α an α-r.e. set A is regular iff for every (or equivalently for one) enumeration f of A (f is an enumeration of A iff $f: \alpha \to A$ is α-recursive, total, 1-1 and onto) we have that $\forall \beta < \alpha \exists \sigma < \alpha (A \cap \beta \subseteq f[\sigma])$ ($f[\sigma] :\equiv f''\sigma$ is the image of the set σ under f).

Although there exist irregular α-r.e. sets for many admissible α (namely those α where α^*, the Σ_1-projection of α, is less than α), the situation is not bad, for Sacks proved in [3] that for all α-r.e. sets A there exists a regular α-r.e. set B of the same α-degree as A (regular set theorem). We can therefore overcome the difficulty of dealing with irregular sets in a priority construction as follows: Instead of performing a construction directly for a given α-r.e. set A, we first choose a representative B of the α-degree of A, which is α-r.e. and in addition regular. Then we apply the construction to B instead of A.

The only unsatisfying point is that this treatment makes the final result nonuniform because all known proofs of the regular set theorem which give the step from A to B contain a nonuniform step. These proofs require us to leave the universe L_α and define from the outside the index of B for a given A, using the extension of A rather than merely the index of A. Sacks asked therefore in

Received August 17, 1976.

[1]This paper was written at M.I.T., where the author was supported by the Deutsche Forschungsgemeinschaft, Bonn.

I would like to emphasize that this paper would not have been possible without the stimulation of Gerald E. Sacks, for which I thank him very much.

[3, Question Q7], whether an α-recursive function can be defined which computes for any given index of an α-r.e. set an index of an α-r.e. regular set of the same α-degree. This question was later repeated by Shore [5], who in the meantime had developed methods which in some cases eliminate nonuniform steps in the priority construction itself.

We give here a positive answer to this question in Theorem 1. We further prove in Theorem 2 that in fact a single natural number n_0 can be found such that for every admissible α the α-recursive function $\{n_0\}$ does the desired work on α. This might be a bit surprising, because deeper theorems which work uniformly for every α are a rare species—the only other member known at present seems to be Shore's uniform solution of Post's problem [6]. As a corollary of Theorem 2 we get a uniform simple set theorem. The application of Theorem 2 to Shore's proof of the splitting theorem yields a relatively uniform version of the splitting theorem.

§0. Preliminaries. A function $f : \alpha \to \alpha$ is called α-recursive if the graph of f is α-r.e. (i.e. $\Sigma_1 L_\alpha$). α^* (the Σ_1 projection of α) is the least ordinal $\delta \leq \alpha$ such that a total α-recursive 1-1 function $P : \alpha \to \alpha^*$ exists. Since α^* is at the same time the least ordinal δ such that an α-r.e. $A \subseteq \delta$ exists with $A \notin L_\alpha$, irregular α-r.e. sets exist iff $\alpha^* < \alpha$. A set $K \subseteq L_\alpha$ is called α-finite if $K \in L_\alpha$. From some fixed universal $\Sigma_1 L_\alpha$ set U we get an indexing $(W_e)_{e \in \alpha}$ for α-r.e. sets.

For sets $A, B \subseteq \alpha$ we say that A is α-recursive in B ($A \leq_\alpha B$) if we have for some e_0, e_1 (which we call the indices of the reduction procedure) that for every $K \in L_\alpha$

$$K \subseteq A \leftrightarrow \exists M, N \in L_\alpha(\langle K, M, N\rangle \in W_{e_0} \wedge M \subseteq B \wedge N \subseteq \alpha - B),$$

$$K \subseteq \alpha - A \leftrightarrow \exists M, N \in L_\alpha(\langle K, M, N\rangle \in W_{e_1} \wedge M \subseteq B \wedge N \subseteq \alpha - B).$$

Observe that we have to check only the second equivalence if A is $\Sigma_1 L_\alpha$.

§1. The uniform regular set theorem for a fixed admissible α. We are going to discuss the nonuniform proof of the regular set theorem first in order to make the problem clear (we use the simplified version of this proof due to Simpson [4]).

Let $f : \alpha \to A$ be an enumeration of an irregular α-r.e. set A. Define the deficiency set D of f by

$$D = \{x \mid \exists y > x(f(y) < f(x))\}.$$

D is then again α-r.e. and in addition regular. If A happens to be a subset of α^* we can see easily:

(1) $A \leq_\alpha D$ because for a given $z < \alpha^*$ we take $x \in \alpha - D$ such that $f(x) > z$ and may then reduce the Π_1 statement $z \notin f[\alpha]$ to the equivalent Σ_1 statement $z \notin f[x]$. We can always find the x for $z < \alpha^*$ such that $x \in \alpha - D$ and $f(x) > z$, because $z \cap A$ is α-r.e. and bounded below α^*, therefore α-finite. Thus $z \cap A \subseteq f[x_0]$ for some x_0 and we may take the $x \geq x_0$ such that $f(x)$ is minimal, which is of course in $\alpha - D$.

(2) $D \leq_\alpha A$, because $K \subseteq \alpha - D \leftrightarrow \bigcup_{x \in K}(f(x) - f[x]) \subseteq \alpha - A$.

If we are not so fortunate as to have that $A \subseteq \alpha^*$ we have to consider in (1) elements $z \geq \alpha^*$ as well and we may not find the desired $x \in \alpha - D$ with $f(x) > z$, because $z \cap A$ need not be α-finite. Therefore the given $A \subseteq \alpha$ is first projected by some α-recursive projection $P : \alpha \to \alpha^*$ into the α-r.e. set $\tilde{A} := P[A]$. Then we take an enumeration $\tilde{f}$ of $\tilde{A}$ and define the desired regular set to be the deficiency set of $\tilde{f}$. But this approach tends to conflict with (2): In order to verify now that $K \subseteq \alpha - D$ we would like to ask whether $P^{-1}[H] \subseteq \alpha - A$ holds for $H := \bigcup_{x \in K}(f(x) - f[x])$. But in order to compute $P^{-1}[H]$ recursively in A we have to compute $H \cap \operatorname{Rg} P$ recursively in A. For this we need that $\operatorname{Rg} P \leq_\alpha A$. Since P cannot be chosen such that $\operatorname{Rg} P$ is α-recursive, we are forced to take for every given A a different projection P_A such that $\operatorname{Rg} P_A \leq_\alpha A$. The canonical way to define such a projection P_A is to look from the outside at the finished enumeration of A and choose the minimal γ_0 such that the enumeration of $\gamma_0 \cap A$ required unboundedly many steps. We then get immediately a total projection $g : \alpha \to \gamma_0 \cap A$ and together with an α-finite 1-1 onto map $h : \gamma_0 \to \alpha^*$ we may define $P_A := h \cdot g$ and as $\operatorname{Rg} P_A = h[\gamma_0 \cap A]$ we have $\operatorname{Rg} P_A \leq_\alpha A$. The choice of γ_0 is in an essential way nonuniform, because there is no hope of computing γ_0 α-recursively from an index of A.

Sacks came up with a very stimulating and natural idea to overcome this difficulty: Define the desired regular set D as an effective disjoint union $D = \{\langle \gamma, x \rangle \mid x \in D_\gamma\}$ of deficiency sets D_γ for every $\gamma < \alpha$, such that every D_γ is a guess at the correct γ_0. So every D_γ should be defined as above, using instead of g a projection g_γ of an initial segment of α onto $\gamma \cap A$. Since g_{γ_0} is the total projection $g : \alpha \to \gamma_0 \cap A$ which was used before, we have the correct deficiency set as the component D_{γ_0} in D. Since we are allowed to use γ_0 as a parameter in the reduction procedure we have then $A \leq_\alpha D$. Unfortunately with this approach one runs into serious difficulties proving that D is regular. Though every single D_γ is regular, the set D might not be regular because every component D_γ uses a different projection.

The following uniform proof of the regular set theorem leaves the idea of embedding the nonuniform proof in a uniform construction behind and is based on an intrinsic uniform proof strategy (which results in getting in addition uniformness with respect to the reduction procedures). We forget γ_0 and take a fixed projection $P : \alpha \to \alpha^*$. Returning to the previous discussion of the nonuniform proof one wanted to compute $\operatorname{Rg} P$ recursively in A. But instead of making $\operatorname{Rg} P$ recursive in A (which causes the nonuniformness) we change the question asked about $\operatorname{Rg} P$. Whereas $\operatorname{Rg} P$ is nonrecursive, $\operatorname{Rg}(P \upharpoonright \gamma)$ is recursive (with γ as additional argument). Therefore we decompose the global deficiency set of $P \cdot f$ into local components D_γ, each of which codes essentially only $A \cap \gamma$ but does not use more of P than $P \upharpoonright \gamma$. The desired regular set $D = \{\langle \gamma, x \rangle \mid x \in D_\gamma\}$ is therefore recursive in A. On the other hand "$z \notin A$" is a "local" property of A and therefore may be recovered from local components D_γ if $\gamma > z$.

THEOREM 1. *Assume α is admissible. Then there is an α-recursive function r such that for all $e \in \alpha$ the α-r.e. set $W_{r(e)}$ is regular and of the same α-degree as W_e. Furthermore the indices of the reduction procedures between W_e and $W_{r(e)}$ can be computed α-recursively from e as well.*

PROOF. Let $P : \alpha \to \alpha^*$ be an α-recursive projection. From a given set W_e we first go to the nonempty set $M := \{0\} \cup \{x+1 \mid x \in W_e\}$ and then to the unbounded set $A := \{\eta \mid K_\eta \cap M \neq \emptyset\}$. For this we use a total map $\eta \to K_\eta$ from α onto L_α. For later use in Theorem 2 we assume that the map is defined uniformly by a parameter-free Δ_1-formula for all admissible α.

We take then an enumeration $f : \alpha \to A$ of A and define the desired regular set as

$$D := \{\langle \gamma, x \rangle \mid \exists y > x (f(y) < \gamma \wedge P(f(y)) < P(f(x)))\}.$$

It is obvious that an index $r(e)$ for D can be computed α-recursively from e, using the index for P as the only parameter.

(a) $A \leq_\alpha D$. Let $z \in \alpha$ be given. $K := (P(z)+1) \cap P[A]$ is α-finite and therefore $P^{-1}[K] \subseteq f[x_0]$ for some x_0. Take $y \geq x_0$ such that $P(f(y))$ is minimal. Using this y as witness for the right side we get "$\rightarrow$" of the following equivalence:

$$z \notin A \leftrightarrow \exists y (\langle z+1, y \rangle \in \alpha - D \wedge P(f(y)) > P(z) \wedge z \notin f[y]).$$

For "$\leftarrow$" assume for a contradiction that $z = f(y_0)$ for some y_0 and that y is witness for the right side. This cannot be because the properties of y_0 imply then that $\langle z+1, y \rangle \in D$.

The set A has by construction the property that for any $B \subseteq \alpha$: A is weakly α-recursive in B iff A is α-recursive in B. Thus we have proved that $A \leq_\alpha D$.

(b) *D is regular over α.* We show that for every $\beta < \gamma$, $D \cap \{\langle \gamma, x \rangle \mid \gamma < \beta \wedge x < \beta\} \in L_\alpha$. Choose $\gamma_1 \leq \beta$ minimal such that $\exists y \geq \beta (f(y) < \gamma_1)$ (γ_1 need not exist, but this case is trivial). If $\gamma_1 < \beta$ take $y_1 \geq \beta$ such that $f(y_1) < \gamma_1$ and $P(f(y_1))$ is minimal. If a γ_2 exists such that $\gamma_1 < \gamma_2 < \beta$ and $\exists y > \beta$ $(f(y) < \gamma_2 \wedge P(f(y)) < P(f(y_1)))$ then choose the least such. Choose y_2 such that $f(y_2) < \gamma_2$ and $P(f(y_2))$ is as small as possible. Continue this construction as long as possible. Since $P(f(y_i))$ is descending the construction has only finitely many steps. We arrive at sequences $\gamma_1, \ldots, \gamma_n$; $y_1, \ldots, y_n$ such that for all $\gamma < \beta$: If $\gamma_i \leq \gamma < \gamma_{i+1}$ then $\min\{P(f(y)) \mid y \geq \beta \wedge f(y) < \gamma\} = P(f(y_i))$ (define $\gamma_{n+1} := \beta$ for completeness). $D \cap \beta \times \beta$ is then α-finite because if $\gamma, x \in \beta$ and $\langle \gamma, x \rangle \in D$, we can either find a witness $y < \beta$ with $y > x$, $f(y) < \gamma$, $P(f(y)) < P(f(x))$ or $\min\{P(f(y)) \mid y \geq \beta \wedge f(y) < \gamma\} < P(f(x))$ in which case we only have to look at the value $P(f(y_i))$ where $\gamma_i \leq \gamma < \gamma_{i+1}$.

(c) $D \leq_\alpha A$. It is obvious that

$$\langle \gamma, y \rangle \in \alpha - D \leftrightarrow \{z \in \gamma \mid P(z) < P(f(y))\} - f[y] \subseteq \alpha - A,$$

therefore

$$K \subseteq \alpha - D \leftrightarrow \bigcup_{\langle \gamma, y \rangle \in K} (\{z \in \gamma \mid P(z) < P(f(y))\} - f[y]) \subseteq \alpha - A$$

where the union $\bigcup_{\langle \gamma, y \rangle \in K} \cdots$ is of course α-finite.

Remarks. (1) This theorem works as well for admissible structures $\langle L_\alpha, C\rangle$ with C regular over L_α.

(2) The additional uniformness with respect to the indices for the reduction procedures shows that the regular set theorem is in fact an "α-effective statement": The regular set theorem claims for every e the existence of three ordinals: an index e' for a regular set $W_{e'}$ and indices for the two reduction procedures $W_e \leq_\alpha W_{e'}$ and $W_{e'} \leq_\alpha W_e$. Theorem 1 shows that witnesses for all three existential quantifiers can be computed α-recursively from e.

§2. The uniform regular set theorem uniformly for all admissible α. Here we want to get rid of the parameter of the projection $P: \alpha \to \alpha^*$ which was used in the definition of the function r in Theorem 1. We are going to do this by using Σ_1-Skolem functions. Proofs for the following easy facts about the fine structure of L can be found in Devlin [1].

For any α one can define uniformly by a Σ_1-definition over L_α without parameters a Σ_1-Skolem function h. h is a partial function $\omega \times L_\alpha \to L_\alpha$ which generates Σ_1-Skolem hulls. We need the following property of h: If $X \subseteq L_\alpha$ is transitive and closed under pairing, we have $h[\omega \times X] = L_\beta$ for some $\beta \leq \alpha$ where $L_\beta \prec_{\Sigma_1} L_\alpha$. Furthermore there is no $\gamma < \beta$ such that $\omega \times X \subseteq L_\gamma \prec_{\Sigma_1} L_\alpha$, due to the parameter-free Σ_1-definition of h.

Define $\tilde{L}_\delta$ to be the closure of $L_\delta \cup \{\delta\}$ under pairing. The function $\delta \mapsto \tilde{L}_\delta$ has a parameter-free Σ_1-definition over L_α and we have $\tilde{L}_\delta \subseteq L_{\delta+\omega}$, $\tilde{L}_\delta \in L_{\delta+\omega+1}$. Every $\tilde{L}_\delta$ has the properties required above for X and therefore $h[\omega \times \tilde{L}_\delta]$ is "the next Σ_1-substructure L_β after L_δ", being equal to L_α if there exists no $\beta < \alpha$ such that $\delta < \beta$ and $L_\beta \prec_{\Sigma_1} L_\alpha$.

Inverting these Σ_1-Skolem hull constructions we get Σ_1-projections: Define $R(\delta, x, y) :\equiv (y = \langle y_0, y_1\rangle \in \tilde{L}_\delta \wedge h(y_0, y_1) = x)$. R has a parameter-free $\Sigma_1 L_\alpha$-definition and therefore we can define a Σ_1-uniformization $P: \alpha \times \alpha \to \alpha$ of R by a parameter-free formula. It follows that for all limit ordinals δ, $P(\delta, \cdot)$ maps $h[\omega \times \tilde{L}_\delta]$ 1-1 into $\tilde{L}_\delta$. In particular if there is no $\delta' > \delta$ such that $\delta' < \alpha$ and $L_{\delta'} \prec_{\Sigma_1} L_\alpha$, we have that $P(\delta, \cdot)$ maps α 1-1 into $\tilde{L}_\delta$. In the case $\alpha^* < \alpha$ we can always find a limit $\delta \geq \alpha^*$ with these properties. We are going to use $P(\delta_0, \cdot)$, where δ_0 is the least δ such that $\operatorname{dom} P(\delta, \cdot) = \alpha$, as the projection in the deficiency set. Though $P(\delta_0, \cdot)$ projects only into $\tilde{L}_{\delta_0}$, we get a projection into α^* by using that α^* is the cardinality of $\tilde{L}_{\delta_0}$ in L_α. Since we do not want to use δ_0 or α^* as parameters in the definition of the deficiency set, we develop for every δ a deficiency set D_δ similar to that in Theorem 1 which uses $P(\delta, \cdot)$ together with an α-finite 1-1 map of $\tilde{L}_\delta$ into the guessed α-cardinality of $\tilde{L}_\delta$ as the projection. We define the desired regular set D as the effective disjoint union of the sets D_δ. Because every D_δ uses a different projection, we have to take precautions in order to keep D regular and recursive in W_e. The idea is to put some restrictions on the definition of every D_δ, which will not harm D_{δ_0} but will deform the other sets D_δ in such a way that we can handle them easily. This is done by using the fact that for every $\delta > \delta_0$ we eventually find a witness for $\neg L_\delta \prec_{\Sigma_1} L_\alpha$ and that for $\delta < \delta_0$, $\operatorname{dom} P(\delta, \cdot) \subseteq L_{\delta_0}$.

THEOREM 2. *There exists a natural number n_0 with the following property: For every admissible $\alpha \geq \omega$ the α-recursive function $\{n_0\}$ is total and for every $e \in \alpha$, $W_{\{n_0\}(e)}$ is a regular set such that $W_{\{n_0\}(e)} =_\alpha W_e$.*

PROOF. We construct a parameter-free Σ_1-formula $\Psi(u, v)$ such that for every admissible $\alpha \geq \omega$ and every $e \in \alpha$ the α-r.e. set $\{v \in \alpha \mid L_\alpha \models \Psi(e, v)\}$ is regular and has the same α-degree as W_e. The desired index n_0 is then essentially a code for Ψ.

From a given α-r.e. set W_e we proceed as in Theorem 1 to an enumeration $f : \alpha \to A$ with index $\tilde{e}$ where $A =_\alpha W_e$. The following regular α-r.e. set D is then of the same α-degree as A:

$$\begin{aligned} D := \{\langle \delta, \gamma, x\rangle \mid & (\delta = \gamma = 0 \wedge \exists y > x(f(y) < f(x))) \vee (\delta \text{ limit} \wedge \tilde{L}_\delta \in L_x \\ & \wedge \exists \xi g \in L_x(L_x \models [g : \tilde{L}_\delta \to \xi \text{ 1-1} \wedge \neg \exists \xi' g'(\xi' < \xi \wedge g' : \tilde{L}_\delta \to \xi' \text{ 1-1})] \\ & \wedge \forall g' \in L_x(g' : L_\delta \to \xi \text{ 1-1} \to g' \leqslant_\alpha g) \\ & \wedge \exists y \exists u (y > x \wedge (\exists v \in \delta(v \text{ limit}) \\ & \to \forall w \in L_\delta(L_y \models \Phi(w) \to L_\delta \models \Phi(w))) \\ & \wedge L_y \models [\neg \exists \xi' g'(\xi' < \xi \wedge g' : \tilde{L}_\delta \to \xi' \text{ 1-1})] \\ & \wedge f(y) < \gamma \wedge \exists x' y' e' \in \tilde{L}_\delta(f(u) < \gamma \wedge P^u(\delta, \tilde{e}) \simeq e' \wedge P^u(\delta, f(x)) \simeq x' \\ & \wedge P^u(\delta, f(y)) \simeq y' \wedge g(y') < g(x')))))\}. \end{aligned}$$

The formula Φ which occurs in the definition is a parameter-free Σ_1-formula with the property that $L_\delta <_{\Sigma_1} L_\alpha \leftrightarrow \forall w \in L_\delta \ (L_\alpha \models \Phi(w) \to L_\delta \models \Phi(w))$. Φ can be defined easily with the help of the Σ_1-Skolem function h.

We further used a wellordering $<_\alpha$ of L_α, which can be defined uniformly (parameter-free) Δ_1 over L_α. We assume that $<_\alpha$ has the property: $\forall \beta \gamma \in \alpha(\beta < \gamma \wedge x \in L_\beta \wedge y \in L_\gamma - L_\beta \to x <_\alpha y)$.

The statement $P^u(\delta, x) \simeq x'$ is an abbreviation for $L_u \models \Phi_0(\delta, x) \simeq x'$ where Φ_0 is the Σ_1-formula defining P.

Those features of the set D which we did not explain before are built in for the sake of the case $\alpha = \alpha^*$. Whereas for $\alpha^* < \alpha$ the set S of Σ_1-substructures L_β of L_α is α-finite, we have for $\alpha^* = \alpha$ that S is unbounded in α and not α-r.e. Therefore the sets D_δ which use $P(\delta, \cdot)$ as a projection are too unstable to code A in a nice way. We added therefore as the set D_0 the deficiency set of f to D. A further precaution insures that the rest of D does not get too strong in the case $\alpha^* = \alpha$: In general we cannot decide recursively in A in this case whether for some δ, z $P(\delta, z)\downarrow$ (converges). In order to avoid brooding too long over this decision, if A happens to be regular we project only those z such that $P(\delta, z)$ converges in a given time. The disadvantage of this limitation is that we have to argue more carefully in the case $\alpha^* < \alpha$ of the proof.

First we introduce some abbreviations for the proof. An α-recursive set M is defined by $\langle \delta, \gamma, x\rangle \in M \leftrightarrow \tilde{L}_\delta \in L_x \wedge L_x \models \exists g \xi \ (g : \tilde{L}_\delta \to \xi \text{ 1-1})$.

An α-recursive function H with dom $H = M$ is given by

$$\begin{aligned} H(\langle \delta, \gamma, x\rangle) = \langle \xi, g\rangle \leftrightarrow L_x \models & [g : \tilde{L}_\delta \to \xi \text{ 1-1} \wedge \neg \exists \xi' g'(\xi' < \xi \wedge g' : \tilde{L}_\delta \to \xi' \text{ 1-1})] \\ & \wedge \forall g' \in L_x(g' : \tilde{L}_\delta \to \xi \text{ 1-1} \to g \leqslant_\alpha g'). \end{aligned}$$

We write $F(y, u, \delta, \gamma, x, \langle g, \xi \rangle)$ for the formula which follows the quantifiers $\exists y \exists u$ in the definition of D. Observe that "$\langle \delta, \gamma, x \rangle \in \operatorname{dom} H \wedge \neg F(y, u, \delta, \gamma, x, H(\langle \delta, \gamma, x \rangle))$" can be expressed by a Σ_1-formula over L_α.

If $\langle \delta, \gamma, x \rangle \in D$ because of the second part of the definition of D, we call those y, u witnesses for $\langle \delta, \gamma, x \rangle \in D$ which, for the appropriate g, ξ, satisfy $F(y, u, \delta, \gamma, x, \langle g, \xi \rangle)$.

For the case $\alpha^* < \alpha$ we write δ_0 for the least limit ordinal δ such that $\operatorname{dom} P(\delta, \cdot) = \alpha$. Observe that $L_{\delta_0} <_{\Sigma_1} L_\alpha$ if $\delta_0 > \omega$. We further write $\hat{x}$ for the least x such that $\tilde{L}_{\delta_0} \in L_x \wedge L_x \models [\exists g : \tilde{L}_{\delta_0} \to \alpha^* \text{ 1-1}]$ and $\hat{g}$ for the $<_\alpha$-minimal g with $L_x \models [g : \tilde{L}_{\delta_0} \to \alpha^* \text{ 1-1}]$.

(1) $\alpha^* < \alpha$. (a) $A \le_{w\alpha} D$: We have

$$\begin{aligned} z \notin A \leftrightarrow \exists \gamma x u (x \ge \hat{x} \wedge z < \gamma \wedge f(u) < \gamma \\ \wedge P^u(\delta_0, \bar{e}) \downarrow \wedge P^u(\delta_0, f(x)) \downarrow \wedge P^u(\delta_0, z) \downarrow \wedge \\ \hat{g}(P(\delta_0, z)) < \hat{g}(P(\delta_0, f(x))) \wedge \neg z \in f[x] \wedge \langle \delta_0, \gamma, x \rangle \in \alpha - D). \end{aligned}$$

"$\to$": $\tilde{z} := \hat{g}(P(\delta_0, z))$. Then $K := \hat{g} \cdot P(\delta_0, \cdot)[A] \cap \tilde{z} + 1$ is α-finite because $\tilde{z} < \alpha^*$. This implies that $K_1 := (\hat{g} \cdot P(\delta_0, \cdot))^{-1}[K]$ is α-finite and a subset of A. Therefore we find $y_0 \ge \hat{x}$ such that $K_1 \subseteq f[y_0]$. Choose $x_0 \ge y_0$ such that the value $\hat{g}(P(\delta_0, f(x_0)))$ is as small as possible. By definition of y_0 we have $\tilde{z} < g(P(\delta_0, f(x_0)))$. We take then some u_0, γ_0 such that $f(u_0) < \gamma_0$, $z < \gamma_0$ and $P^{u_0}(\delta_0, \bar{e}) \downarrow$, $P^{u_0}(\delta_0, f(x_0)) \downarrow$, $P^{u_0}(\delta_0, z) \downarrow$. It is obvious that γ_0, x_0, u_0 satisfy the right side of the equivalence.

"$\leftarrow$": Assume that the right side holds for γ, x, u and $z = f(y)$ for some y. It follows that y, u are witnesses for $\langle \delta_0, \gamma, x \rangle \in D$, a contradiction.

(b) *D is regular over L_α*: Fix $\beta < \alpha$. We want to show that $\{\langle \delta, \gamma, x \rangle \mid \delta, \gamma, x < \beta\} \cap D \in L_\alpha$. Define $M_1 := \{\langle \delta, \gamma, x \rangle \mid \delta = \gamma = 0 \wedge x < \beta\}$. Take $y_0 \ge \beta$ such that $f(y_0)$ is as small as possible. Then $\langle 0, 0, x \rangle \in M_1$ gets into D iff there exists $y \le y_0$ such that $y > x \wedge f(y) < f(x)$. It follows that $M_1 \cap D \in L_\alpha$. Define

$$\tilde{M} := \{\langle \delta, \gamma, x \rangle \mid \delta, \gamma, x < \beta \wedge \delta \text{ limit} \wedge \tilde{L}_\delta \in L_x \wedge L_x \models \exists g \xi (g : \tilde{L}_\delta \to \xi \text{ 1-1}).$$

$\tilde{M}$ is α-finite.

Define $M_2 := \{\langle \delta, \gamma, x \rangle \in \tilde{M} \mid \delta > \delta_0\}$. Choose v_0 such that

$$\forall \delta < \beta (\delta > \delta_0 \to \exists w \in L_\delta (L_{v_0} \models \Phi(w) \wedge L_\delta \models \neg \Phi(w))).$$

Then for elements of M_2 we only have to consider witnesses y where $y < v_0$. Choose $z_0 > e, f[\beta], \beta$. Take u_0 such that $\forall \delta < \beta \forall z < z_0 (\delta > \delta_0 \to P^{u_0}(\delta, z) \downarrow)$. Take $u_1 \ge u_0$ such that $f(u_1)$ is as small as possible. Then it is obvious that if any witness u for an element of M_2 exists, we can find such a witness $u \le u_1$. Since we have found a priori bounds on witnesses y, u which put elements of M_2 into D, it is obvious that $M_2 \cap D \in L_\alpha$.

$M_3 := \{\langle \delta, \gamma, x \rangle \in \tilde{M} \mid \delta = \delta_0 \wedge x < \hat{x}\}$. Then $\hat{x}$ is an a priori bound on witnesses y. Take z_0 as before and choose u_0 such that $\forall z < z_0 (P^{u_0}(\delta_0, z) \downarrow)$. Take u_1 such that $u_1 \ge u_0$ and $f(u_1)$ is minimal. Then u_1 is an a priori bound on u.

$M_4 := \{\langle\delta,\gamma,x\rangle \in \tilde{M} \mid \delta = \delta_0 \wedge x \geq \hat{x}\}$. A bound u_1 on witnesses u is defined just as for M_3. For witnesses y we argue as in the proof of Theorem 1. Take $\gamma_1 \leq \beta$ as large as possible such that

$$\forall\gamma < \gamma_1 \neg\exists y \geq \beta\exists u(f(y) < \gamma \wedge f(u) < \gamma \wedge P^u(\delta_0, f(y))\downarrow).$$

If $\gamma_1 < \beta$ take $y_1 \geq \beta$ which satisfies $f(y_1) < \gamma_1 \wedge \exists u(f(u) < \gamma_1 \wedge P^u(\delta_0, f(y_1))\downarrow)$ such that the value $\hat{g}(P(\delta_0, f(y_1)))$ is as small as possible. We then look for the minimal $\gamma_2 < \beta$ such that $\gamma_2 > \gamma_1$ and

$$\exists y \geq \beta(f(y) < \gamma_2 \wedge \exists u(f(u) < \gamma_2 \wedge P^u(\delta_0, f(y))\downarrow \wedge \hat{g}(P(\delta_0, f(y))) < \hat{g}(P(\delta_0, f(y_1))))).$$

Again we take y_2 such that $\hat{g}(P(\delta_0, f(y_2)))$ is minimal. Since the sequence $\hat{g}(P(\delta_0, f(y_i)))$ is descending this process stops after getting sequences $\gamma_1,\ldots,\gamma_n; y_1,\ldots,y_n$. Define $\gamma_{n+1} := \beta$. Then for elements $\langle\delta,\gamma,x\rangle$ of M_4 we only have to consider witnesses $u \leq u_1$, $y \leq \beta$ and—if $\gamma_i \leq \gamma < \gamma_{i+1}$—the witness y_i.

$M_5 := \{\langle\delta,\gamma,x\rangle \in \tilde{M} \mid \delta < \delta_0\}$. If y is witness for $\langle\delta,\gamma,x\rangle \in D$ where $\langle\delta,\gamma,x\rangle \in M_5$, then $P(\delta, f(y))\downarrow$, $P(\delta, \tilde{e})\downarrow$ and $L_{\delta_0} <_{\Sigma_1} L_\alpha$. This implies that $y < \delta_0$ and therefore δ_0 is an a priori bound on witnesses y. A bound on witnesses u is given by u_1 such that $u_1 \geq \delta_0$ and $f(u_1)$ is minimal.

(c) $D \leq_\alpha A$: For this part of the proof we analyze the given set K in essentially the same way as we did with the set M in (b). For those parts of K which correspond to the previous sets M_1, M_4 we set up the computation in A which is typical for deficiency sets. For the other parts we again search for a priori bounds. This gives rise to the following computation:

$$\begin{aligned}
K \subseteq \alpha - D \leftrightarrow\ & \exists z z_0(z > \sup K \wedge z_0 > z, e, f[z] \wedge \\
& \exists K_1(K_1 = \{\langle\delta,\gamma,x\rangle \in K \mid \delta = \gamma = 0\} \wedge \\
& \bigcup_{\langle 0,0,x\rangle\in K_1} (f(x) - f[x]) \subseteq \alpha - A) \wedge \exists\tilde{K}(\tilde{K} \\
& = \{\langle\delta,\gamma,x\rangle \in K \mid \delta \text{ limit} \wedge \tilde{L}_\delta \in L_x \wedge \\
& L_x \models \exists g\xi(g : L_\delta \to \xi \text{ 1-1})\} \wedge \exists K_2(K_2 = \{\langle\delta,\gamma,x\rangle \in \tilde{K} \mid \delta > \delta_0\} \wedge \\
& \exists v_0(\forall\delta < z(\delta \text{ limit} \wedge \delta > \delta_0 \to \\
& \exists w \in L_\delta(L_{v_0} \models \Phi(w) \wedge L_\delta \models \neg\Phi(w))) \wedge \\
& \exists u_0 \forall v < z_0 \forall\delta < z((\delta \text{ limit} \wedge \delta > \delta_0 \to P^{u_0}(\delta, v)\downarrow) \wedge \\
& \exists u_1 \geq u_0(f(u_1) - f[u_1] \subseteq \alpha - A \wedge \\
& \forall\langle\delta,\gamma,x\rangle \in K_2 \forall y < v_0 \forall u \leq u_1 \\
& \neg(F(y,u,\delta,\gamma,x,H(\langle\delta,\gamma,x\rangle)))))) \wedge \exists K_3(K_3 = \\
& \{\langle\delta,\gamma,x\rangle \in \tilde{K} \mid \delta = \delta_0 \wedge x < \hat{x}\} \wedge \\
& \exists u_0 \forall v < z_0(P^{u_0}(\delta_0, v)\downarrow \wedge \exists u_1 \geq u_0
\end{aligned}$$

$$(f(u_1) - f[u_1] \subseteq \alpha - A \wedge \forall \langle \delta, \gamma, x\rangle \in K_3 \forall y < \hat{x} \forall u \leq u_1$$
$$\neg F(y, u, \delta, \gamma, x, H(\langle \delta, \gamma, x\rangle)) \wedge$$
$$\exists K_4(K_4 = \{\langle \delta, \gamma, x\rangle \in \tilde{K} \mid \delta = \delta_0 \wedge x \geq \hat{x}\} \wedge$$
$$\bigcup_{\langle \delta,\gamma,x\rangle \in K_4} (\{z \in \gamma \mid \exists u \leq u_1$$
$$(f(u) < \gamma \wedge P^u(\delta_0, \tilde{e})\downarrow \wedge P^u(\delta_0, f(x))\downarrow \wedge P^u(\delta_0, z)\downarrow \wedge$$
$$\hat{g}(P(\delta_0, z)) < \hat{g}(P(\delta_0, f(x))))\} - f[x]) \subseteq \alpha - A)) \wedge \exists K_5(K_5 =$$
$$\{\langle \delta, \gamma, x\rangle \in \tilde{K} \mid \delta < \delta_0\} \wedge \exists u_1 \geq \delta_0(f(u_1) - f[u_1] \subseteq \alpha - A \wedge$$
$$\forall \langle \delta, \gamma, x\rangle \in K_5 \forall y < \delta_0 \forall u \leq u_1 \neg F(y, u, \delta, \gamma, x, H(\langle \delta, \gamma, x\rangle)))))).$$

(2) $\alpha^* = \alpha$. (a) $A \leq_{w\alpha} D$: $z \notin A \leftrightarrow \exists x(f(x) > z \wedge \neg z \in f[x] \wedge \langle 0,0,x\rangle \in \alpha - D)$.

(b) $D \leq_\alpha A$: For $\alpha = \omega$ we have $K \subseteq \omega - D \leftrightarrow \bigcup_{\langle 0,0,x\rangle \in K}(f(x) - f[x]) \subseteq \alpha - A$.

For $\alpha > \omega$ we have (using the regularity of A):

$$K \subseteq \alpha - D \leftrightarrow \bigcup_{\langle 0,0,x\rangle \in K} (f(x) - f[x]) \subseteq \alpha - A \wedge$$
$$\exists z y_0(z > K \wedge z - f[y_0] \subseteq \alpha - A \wedge$$
$$\exists K'(K' = \{\langle \delta, \gamma, x\rangle \in K \mid \delta \text{ limit} \wedge \tilde{L}_\delta \in L_x \wedge$$
$$L_x \models \exists g \xi (g : \tilde{L}_\delta \to \xi \text{ 1-1})\} \wedge$$
$$\forall \langle \delta, \gamma, x\rangle \in K' \forall y < y_0 \forall u < y_0 \neg F(y, u, \delta, \gamma, x, H(\langle \delta, \gamma, x\rangle))))).$$

REMARK. The indices of the preceding reduction procedures can again be computed α-recursively from e for every fixed α. It seems unlikely that one can eliminate those parameters in the reduction procedures which come from α (δ_0 etc.)

§3. Applications. We will discuss applications to inadmissible structures in [2].

An immediate application of Theorem 2 is the following.

UNIFORM SIMPLE SET THEOREM. *There exists a natural number n_1 with the following property: For every admissible $\alpha \geq \omega$ the α-recursive function $\{n_1\}$ is total and for every $e \in \alpha$, $W_{\{n_1\}(e)} =_\alpha W_e$ where $W_{\{n_1\}(e)}$ is regular and simple if W_e is not α-recursive.*

PROOF. An α-r.e. set $B \subseteq \alpha$ is called simple if $\alpha - B$ is unbounded but does not contain an unbounded α-r.e. set (this implies B not α-recursive). It is well known that if $g : \alpha \to D$ is an enumeration of a regular nonrecursive set D we have that $B = \{x \mid \exists y > x(g(y) < g(x))\}$ is simple and of the same α-degree as D (B is automatically regular as well). Since this step from D to B is uniform, we just have to apply this step to the set $D = W_{\{n_0\}(e)}$ which is given by Theorem 2.

For the following application we use Shore's uniform method for splitting regular sets from [5] (which is only nonuniform with respect to the break-up into the cases $\alpha^* < \alpha$ and $\alpha^* = \alpha$) and get a

(RELATIVELY) UNIFORM SPLITTING THEOREM. *There exist natural numbers n, m with the following property: For every admissible α the α-recursive functions* $\{n\}$, $\{m\}$ *are total and for every* $e_1, e_2 \in \alpha$ *we get sets* $A_i :\equiv W_{\{n\}(e_1,e_2,i)}$ *and* $B_i :\equiv W_{\{m\}(e_1,e_2,i)}$ *such that* $A_i \oplus B_i =_\alpha W_{e_1}$, $A_i \cap B_i = \emptyset$, $W_{e_2} \not\leq_\alpha A_i$ *and* $W_{e_2} \not\leq_\alpha B_i$ *if* $W_{e_2} >_\alpha 0$ *where* $i = 0$ *if* $\alpha^* < \alpha$ *and* $i = 1$ *if* $\alpha^* = \alpha$.

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